ON CERTAIN GENERAL PROPERTIES OF THE PHOTON PROPAGATION FUNCTION IN QUANTUM ELECTRODYNAMICS

A. A. ANSEL'M

Leningrad Physico-Technical Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor November 14, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **38**, 1288-1296 (April, 1960)

By considering jointly the spectral representation of the photon Green's function and the renormalizability property, the behavior of the D function is investigated for very large energies and $e^2 = 1/137$, and for very large charges but not too high energies. With an accuracy to within a numerical parameter it was possible to establish the dependence of the D function on charge in the first case, and on energy in the second case.

At present it is still an open question whether the difficulties that occur when quantum field theory is applied to phenomena taking place at small distances indicate an internal inconsistency of the theory or whether they are due to the inadequacy of the existing calculation methods. Although objections have been raised repeatedly against the double-limit proof of the zero-charge, the arguments of Landau and Pomeranchuk (which, however, refer only to electrodynamics) are still quite convincing, even if they have only qualitative character (see, for example, the article by Landau[1]).

In this connection it appears to be of interest to try to investigate the photon Green's function for very large energies, using only the most general features of the theory: positive definiteness of probabilities, renormalizability, gauge invariance, etc. This can be accomplished partially. As will be seen below, the qualitative concepts used in the investigation can be made self-consistent, which leads us to conclude, for example, that it is not possible to prove rigorously the existence of a zero-charge using only renormalizability and the Lehmann expansion. At the same time, although these two conditions do not, of course, determine the propagation function completely, they do allow us to say more than one should have expected about its qualitative properties.

In the first section of this paper we investigate the behavior of the photon Green's function in the region of energies much larger than the critical energy (corresponding to the unphysical pole appearing in the Green's function in the usual calculations). Up to a numerical constant it is possible to establish the dependence of the D function on the charge, and it turns out that the D function either does not depend on the charge at all or is exponentially small. The last case corresponds to the non-logarithmic character of the divergence of the renormalized constant in the exact solution of the equations of quantum electrodynamics and seems to be entirely possible.

In the second section we shall consider a model of quantum electrodynamics for very large values of the renormalized coupling constant. It is possible to find the explicit dependence of the Green's function on the energy in a wide region. It should be noted that these results can be obtained only by making the very critical assumption that the role of the mass term in the Lagrangian does not become too important as the coupling constant increases.

In conclusion we shall discuss the problem of the zero charge in connection with the preceding considerations. It will be noted that our results do not in any way refute the arguments of Landau and Pomeranchuk, who considered the Hamiltonian directly, although our discussion starts from the very beginning with the assumption that there is no zero charge (i.e., that it is possible to use the basic general principles of the theory consistently with a nonvanishing value of the renormalized charge).

1. THE PHOTON GREEN'S FUNCTION FOR VERY LARGE ENERGIES

The renormalized photon propagation function $D_C = d_C/k^2$ is a function of two variables: the dimensionless ratio k^2/m^2 (k is the energy-momentum vector of the particle, and m is the mass of the electron) and the square of the renormalized charge e^2. Gell-Mann and Low showed from the

renormalizability requirement of the theory that the function d_c $(k^2/m^2, e^2)$ can be expressed in terms of a function of a single variable in the asymptotic region $k^2/m^2 \gg 1$:[2]

$$e^2 d_c(k^2/m^2, e^2) = F(k^2 \varphi(q(e^2))/m^2, \quad (1)$$

where F and φ are inverse functions, and $q(e^2)$ nearly coincides with e^2 for small values of the charge:

$$q(e^2) = e^2[1 - (5/9\pi)e^2 + \ldots]. \quad (2)$$

Introducing the functions

$$h(x) = F(e^x), \quad g(x) = \ln \varphi(x),$$
$$\psi(x) = h(-x^{-1}), \quad \chi(x) = -1/g(x), \quad (3)$$

we rewrite (1) in the form

$$e^2 d_c = h[\xi + g(q(e^2))] = \psi[\chi(q(e^2))(1 - \chi(q(e^2))\xi)^{-1}],$$
$$\xi = \ln(k^2/m^2), \quad (4)$$

where h and g, and ψ and χ are inverse functions.

As is known, perturbation theory gives the following expressions for the functions $\chi(x)$ and $\psi(x)$ for small positive x:

$$\chi(x) = x/3\pi, \quad \psi(x) = 3\pi x, \quad 0 < x \ll 1, \quad (5)$$

which leads to the known expression for $e^2 d_c$:[3]

$$e^2 d_c = e^2[1 - (e^2/3\pi)\xi]^{-1}, \quad e^2 \ll 1, \quad e^2 d_c \ll 1. \quad (6)$$

This means that the functions g and h are determined by perturbation theory for the following values of the argument:

$$g(x) = -3\pi/x \text{ for } 0 < x \ll 1,$$
$$h(x) = -3\pi/x \text{ for } x < 0, \ |x| \gg 1. \quad (7)$$

The Green's function d_c can be written in the form of an expansion in terms of the masses:[2,4]

$$d_c(k^2/m^2, e^2) = 1 + k^2 \int_0^\infty d\varkappa^2 \sigma(\varkappa^2, e^2)/(k^2 + \varkappa^2) \quad (8)$$

and is therefore an increasing function of k^2 (or $\xi = \ln(k^2/m^2)$, if we consider space-like $k^2 > 0$):

$$\frac{\partial}{\partial k^2} d_c(k^2/m^2, e^2) = \int_0^\infty d\varkappa^2 \varkappa^2 \sigma(\varkappa^2, e^3)/(k^2 + \varkappa^2)^2 > 0. \quad (9)$$

Since, for values of ξ which go up to $+\infty$ and for small values of e^2 which start from zero, the argument of the function h(x), which is equal to $\xi + g(e^2) \approx \xi - 3\pi/e^2$, varies from $-\infty$ to $+\infty$, the function h(x) is an increasing function of its argument everywhere (see Fig. 1). Two cases are possible:

1) $h[\xi + g(q(e^2))] \to \infty$ for $\xi \to \infty$ (curve *1*),
2) $h[\xi + g(q(e^2))] \to \beta$ for $\xi \to \infty$ (curve *2*),

where β is a numerical constant which is independent of the value of the renormalized charge.

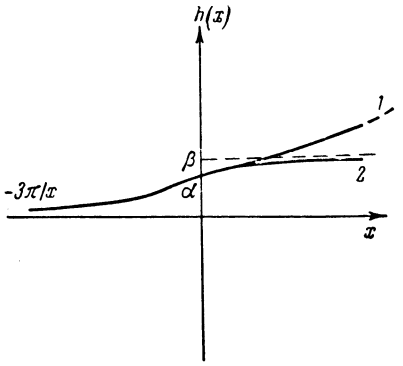

FIG. 1

The behavior of g(x) [the inverse function of h(x)] is shown in Fig. 2. In case 2 the argument of g(x) changes only from zero to β, so that $q(e^2)$ is limited from above by the value $q(e^2) = \beta$. There exists, therefore, a maximal value of the renormalized charge for which quantum electrodynamics is valid: $e^2 \le e_1^2$, $q(e_1^2) = \beta$. We note that then $e_1^2 \le \beta$, since e_1^2 is the value of the function $e_1^2 d_c(k^2/m^2, e_1^2)$ for $k^2 = 0$ and β is the value of the function $e_1^2 d_c(k^2/m^2, e_1^2)$ for $k^2 \to \infty$, and d_c must increase as k^2 increases. This result was obtained by Gell-Mann and Low.[2]

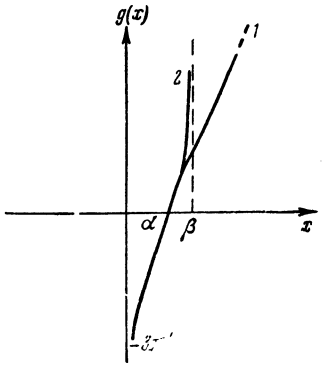

FIG. 2

The behavior of the functions $\psi(x)$ and $\chi(x)$ is shown in Figs. 3 and 4. For $\xi = 1/\chi(e^2)$ (which corresponds to $(e^2/3\pi)\xi = 1$ for small values of the charge) the argument of the function $\psi(x)$ jumps from $+\infty$ to $-\infty$. If the first approximation $\psi(x) = 3\pi x$ is used, this discontinuity represents an unphysical pole in the Green's function ("zero charge"). If we consider the function $\psi(x)$ as shown in Fig. 3, we see that $\psi(x)$ remains continuous for $x = \pm\infty$ [it is equal to α, the root of

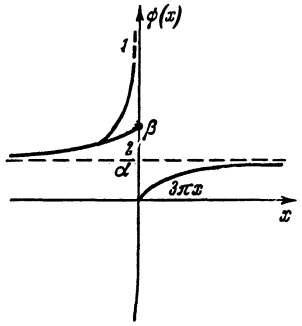

FIG. 3

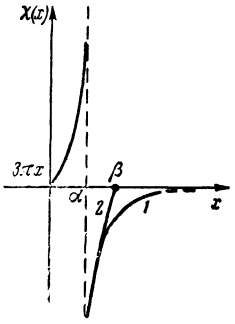

FIG. 4

the function g(x)]. These curves, of course, do not prove in any sense the absence of the zero charge. To decide this question, we must first find out how adequately the figures represent these functions, if the latter are derived directly from a study of the Hamiltonian itself. This problem will be discussed below.

Let us now attempt to obtain some information on the behavior of the Green's function for very large energies $\xi \gg \xi_0$, $\xi_0 = -g(e^2) \approx 3\pi/e^2$. If h(x) were a function in the argument of which small terms could be neglected in comparison with large terms, even if the small terms are not small as compared with unity, we could conclude immediately that for $\xi \gg 3\pi/e^2$, $e^2 d_c$ does not depend on the charge: $e^2 d_c = h(\xi)$. However, h(x) may have exponential character, and then, of course, the dependence on the charge remains in force for arbitrarily large values of ξ.

Let us now try to calculate the value of the ratio $h(x+y)/h(x)$, when $x \gg y$ $(x \to \infty)$ and $y \gtrsim 1$. For this purpose we consider R(y):

$$R(y) = h[\eta + g(e'^2)]/h[\xi + g(e'^2)]$$
$$= \left[1 + p^2 \int_0^\infty dx^2 \sigma(x^2, e'^2)/(p^2 + x^2)\right]$$
$$\times \left[1 + k^2 \int_0^\infty dx^2 \sigma(x^2, e'^2)/(k^2 + x^2)\right]^{-1}, \qquad (10)$$

when $\eta - \xi = y$ remains a finite quantity and η and $\xi \to \infty$; e'^2 is arbitrary. (R(y) is independent of e'^2.) Under these conditions $p^2/k^2 = e^y \equiv \lambda$ is a finite quantity, while $p^2 \to \infty$ and $k^2 \to \infty$. Each of the integrals in the numerator and the denominator of (10) goes to ∞ if $d_c \to \infty$, i.e., we have case 1 (cf. the figures). In case 2 we have $R \to \beta/\beta = 1$. This value of R(y) is included as a special case in the subsequent discussion, and we shall, therefore, assume that the integrals in (10) do not diverge. The R is written in the form

$$R = \lambda F(\lambda)/F(1), \qquad F(\lambda) = \int_0^\infty dx^2 \sigma(x^2, e'^2)/(\lambda k^2 + x^2). \qquad (11)$$

F(λ) and F(1) go to zero for $k^2 \to \infty$. Therefore

$$\frac{F(\lambda)}{F(1)} = \frac{\partial}{\partial k^2} F(\lambda) \Big/ \frac{\partial}{\partial k^2} F(1) = \lambda \frac{\partial F}{\partial \lambda} \Big/ \left(\frac{\partial F}{\partial \lambda}\right)_{\lambda=1}, \qquad (12)$$

or

$$(\partial F/\partial\lambda)_{\lambda=1}/F(1) - \lambda(\partial F/\partial\lambda)/F(\lambda) = 0. \qquad (13)$$

Integrating the last equation, we find*

$$R/\lambda = F(\lambda)/F(1) = \lambda^k, \qquad k = (\partial F/\partial\lambda)_{\lambda=1}/F(1),$$

or

$$R(y) = e^{\nu y}, \quad \nu = 1 - \lim_{k^2\to\infty}\left[k^2 \int_0^\infty \frac{\sigma(x^2, e'^2)\,dx^2}{(k^2+x^2)^2} \Big/ \int_0^\infty \frac{\sigma(x^2, e'^2)\,dx^2}{k^2+x^2}\right]. \qquad (14)$$

We have therefore proved the following property of the function h:

$$h(x+y)/h(x) \to e^{\nu y}, \qquad x \to \infty, \qquad (15)$$

where the numerical parameter ν is given by (14). R(y), and hence ν, do not depend on e'^2, which was included in (10) in an arbitrary way. Nevertheless, ν cannot be calculated by perturbation theory because its applicability requires not only $e'^2 \ll 1$ but also $e'^2 \ln(k^2/m^2) \ll 1$. The function $\nu(k^2)$, which becomes ν in the limit $k^2 \to \infty$, is zero for $e'^2 \ll 1$, $e'^2 \ln(k^2/m^2) \ll 1$ (perturbation theory), depends on k^2 in a complicated manner for $e'^2 \ln(k^2/m^2) \sim 1$, and becomes equal to the above-mentioned number ν for $k^2 \to \infty$.

From (14) we find the following limitations on ν:

$$0 \leqslant \nu \leqslant 1. \qquad (16)$$

If we set

$$h(x) = e^{\nu x} P(x), \qquad (17)$$

we see that the function P(x) has the following property:

$$P(x+y)/P(x) \to 1, \qquad x \to \infty. \qquad (18)$$

Going back to formula (4), we have now

$$e^2 d_c(k^2/m^2, e^2) = \exp\{\nu g(q(e^2))\}(k^2/m^2)^\nu P(\ln(k^2/m^2)),$$
$$\ln(k^2/m^2) \gg g(q(e^2)). \qquad (19)$$

The function g[q(e²)] has for small e^2 the form[5]

$$g(q(e^2)) = -3\pi/q(e^2) - \tfrac{9}{4}\ln q(e^2) + C_1 q(e^2)$$
$$+ C_2[q(e^2)]^2 + \ldots. \qquad (20)$$

*It can be easily shown that the value of F(λ)/F′(1), at which, in fact, the cancellation in the derivation of (13) was carried out, is finite.

We see from (19) that we must take the first two terms of this expansion. Then we have, finally,

$$e^2 d_c(k^2/m^2, e^2) = \exp\{-3\pi\nu/e^2\}(e^2)^{-9\nu/4}(k^2/m^2)^{\nu} P(\ln(k^2/m^2)),$$
$$e^2 \ll 1, \qquad (e^2/3\pi)\ln(k^2/m^2) \gg 1.$$

If we considered the next approximation for the function $q(e^2)$, we would find, according to (2), that this only introduces an unessential constant factor on the right hand side of (20). The energy dependence given by formula (20) is almost trivial and could have been predicted without calculations. The result given above is useful only in that it leads to a connection between the value of the parameter ν with the characteristic integrals (14). Indeed, if $\int \kappa^{-2}\sigma(\kappa^2)\,d\kappa^2$ converges and $\int \sigma(\kappa^2)\,d\kappa^2$ diverges (case 1), the dependence of σ on powers of κ^2 can only have the form $(\kappa^2)^{\nu-1}$ for $0 \leq \nu \leq 1$, which gives at once an energy dependence of the type (20). The case $\nu = 0$ corresponds to a logarithmic divergence of the renormalized coupling constant,

$$Z_3^{-1} = 1 + \int \sigma(\kappa^2)\,d\kappa^2$$

and does indeed occur in the perturbation theory, as was mentioned above. For a finite renormalization (case 2), as in the case of a logarithmic divergence, $e^2 d_c$ does not depend on the charge in the region of large energies. As is seen from (20), the quantity $e^2 d_c$, as a function of the charge, is exponentially small for $\nu > 0$, and can therefore not be expanded in a perturbation series. Here the circumstance that the derivation of (20) was based on the property (1), which was obtained in reference 2 precisely with the help of perturbation theory, is probably not essential for the validity of (20), since the renormalizability property reflects only the most general aspects of the theory. If $\nu \neq 0$, Z_3^{-1} diverges like $\Lambda^{2\nu}$ (Λ is the cut-off momentum).

It should be emphasized that, although the case $\nu = 0$ is intuitively regarded as the most probable, there do not seem to exist any special reasons for excluding values of $\nu \neq 0$.

In the following section we shall discuss quantum electrodynamics for large values of the renormalized coupling constant, where the cases $\nu = 0$ and $\nu \neq 0$ both seem to be perfectly acceptable, even if they lead to D functions of completely different qualitative behavior.

2. THE PHOTON GREEN'S FUNCTION FOR LARGE VALUES OF THE RENORMALIZED COUPLING CONSTANT

Since we are not able to construct a quantitative theory of strong coupling, it is of interest to investigate the general properties of such a theory, even if only on the example of electrodynamics. We shall again start from the renormalizability property (4), which we rewrite in the form

$$q(e^2) = h[g(e^2 d_c) - \xi]. \tag{21}$$

If the magnitude of the charge e^2 becomes much larger than α, the root of the function $g(x)$ (see Fig. 2), and goes to infinity in case 1 or to e_1^2 [$q(e_1^2) = \beta$, $g(\beta) = \infty$] in case 2, the quantities $g(e^2 d_c)$ and $g[q(e^2)]$ in formulas (21) and (4) increase beyond limit. If at the same time the energy does not become too large, so that the inequalities $g(e^2 d_c) \gg \xi$ or $g[q(e^2)] \gg \xi$ are fulfilled, we can use the property (15) of the function $h(x)$ and write

$$d_c(k^2/m^2, e^2) = e^{-2} q(e^2)(k^2/m^2)^{\nu} = d_c(1, e^2)(k^2/m^2)^{\nu}. \tag{22}$$

The quantity $q(e^2)/e^2$ equals $d_c(1, e^2)$, i.e., the value of the asymptotic Green's function for $k^2 = m^2$. It follows from the derivation that we have neglected those terms in (22) which are small in comparison with $d_c(1, e^2)$, although they may be greater than, or of the order of, unity, if only $d_c(1, e^2)$ is large.

In the discussion above we have made one very critical assumption. The criterion for the applicability of (22) is given by the inequalities $\xi \ll g(e^2 d_c)$ or $\xi \ll g[q(e^2)]$, which limit the quantity k^2/m^2 from above. On the other hand, formulas (4) and (20) have asymptotic character, i.e., terms which vanish as k^2/m^2 increases are omitted in these expressions. It is clear that, if these terms grow very fast with increasing e^2 (roughly speaking, faster than $\sim \exp[g(e^2)]$), it is impossible to find a region of values of k^2/m^2 which are large enough that these terms can be neglected, but still sufficiently small to satisfy the above-mentioned inequalities. We shall assume that this difficulty does not occur, basing our supposition on the rather natural hypothesis that the mass term in the Lagrangian cannot have too much importance for very large values of the renormalized charge.

Formula (22) determines the dependence of d_c on energy in a rather wide region. The exact expression (not the asymptotic one) for d_c should have branch points at the threshold values of k^2 which are multiples of the square of the mass. In the asymptotic theory $m^2 \to 0$, and all branch points converge to the point $k^2 = 0$. In this case the character of the branching is $(k^2)^{\nu}$. This throws, perhaps, some light on the meaning of the parameter ν.

It is easy to find the spectral function $\sigma(\kappa^2, e^2)$ corresponding to expression (22). Since

$$\varkappa^2\sigma(\varkappa^2, e^2) = \pi^{-1}\operatorname{Im} d_c(-\varkappa^2/m^2, e^2), \quad (23)$$

we find, after substitution of (22),

$$\varkappa^2\sigma(\varkappa^2, e^2) = \pi^{-1} d_c(1, e^2)(\varkappa^2/m^2)^{\nu}\sin\pi\nu. \quad (24)$$

Equation (24) is valid with an accuracy up to terms which are small in comparison with $d_c(1, e^2)$ if $\kappa^2/m^2 \gg 1$, but

$$\ln(\varkappa^2/m^2) \ll g(e^2 d_c),\ g(q(e^2)).$$

In the case $\nu = 0, 1$, $d_c(k^2/m^2, e^2) = d_c(1, e^2)$ with the same accuracy, and $\sigma/d_c(1, e^2) \to 0$ with increasing e^2. If, in this case, the renormalization is finite, $d_c(1, e^2) \to \beta/e_1^2$ and $\sigma(\kappa^2, e^2)$ itself goes to zero.

Let us consider the case $\nu = 0$ in more detail. Since the basic term in σ which is proportional to $d_c(1, e^2)$ vanishes, it is desirable to determine the next term of the expansion. It follows from formula (23) and the renormalizability condition (4) that the dependence of σ on κ^2 and e^2 in the asymptotic region has the form

$$\varkappa^2\sigma(\varkappa^2, e^2) = e^{-2} f[\ln(\varkappa^2/m^2) + g(q(e^2))], \quad (25)$$

where $f[x]$ is some function which, in the case $\nu = 0$, satisfies condition (18). The asymptotic expression for $d_c(k^2/m^2, e^2)$ for $\nu = 0$ consists of a term independent of k^2/m^2, $d_c(1, e^2)$, and a term which is an increasing function of it. The latter is small in the region under consideration, since for $\nu = 0$

$$d_c(k^2/m^2, e^2)/d_c(1, e^2) \to 1. \quad (26)$$

We can easily obtain the increasing part of d_c, starting from (25). We have

$$d_c\left(\frac{k^2}{m^2}, e^2\right) \approx k^2 \int\limits_{\sim m^2}^{\infty} d\varkappa^2 e^{-2} f[\ln(\varkappa^2/m^2) + g(q(e^2))]/\varkappa^2(\varkappa^2 + k^2).$$

Since the integral converges, the important values of κ^2 are $\kappa^2 \lesssim k^2$, i.e.,

$$\ln(\varkappa^2/m^2) \leqslant \ln(k^2/m^2) \ll g(q).$$

Inasmuch as f satisfies property (18), we obtain at once

$$d_c(k^2/m^2, e^2) \approx e^{-2} f[g(q(e^2))]\ln(k^2/m^2) \quad (27)$$

or, adding the constant term to d_c,

$$d_c(k^2/m^2, e^2) = d_c(1, e^2) + e^{-2} f[g(q(e^2))]\ln(k^2/m^2). \quad (28)$$

Using condition (26), we rewrite (28) in the form

$$d_c(k^2/m^2, e^2) = d_c(1, e^2)[1 + \Delta(e^2)\ln(k^2/m^2)], \quad (29)$$

$$\Delta(e^2) = f[g(q(e^2))]/e^2 d_c(1, e^2) \to 0. \quad (30)$$

Now the $\sigma(\kappa^2, e^2)$ corresponding to (29) is

$$\varkappa^2\sigma(\varkappa^2, e^2) = d_c(1, e^2)\Delta(e^2). \quad (31)$$

Equation (30) is, of course, equivalent to the above-mentioned condition $\sigma/d_c(1, e^2) \to 0$ for increasing e^2. If the renormalization is finite, $q(e^2) \to \beta$ and $d_c(1, e^2) \to \beta/e_1^2$ for $e^2 \to e_1^2$. Here $\sigma(\kappa^2, e^2)$ goes to zero like $\Delta(e^2)$, which in its own right must vanish faster than $[g(q)]^{-1}$, as can be seen from the definition (30) and the convergence of the integral

$$\int_{\sim}^{\infty} \sigma(\varkappa^2)\, d\varkappa^2 = e^{-2}\int_{\sim}^{\infty} f(x)\, dx.$$

If the renormalization is infinite, it is easily shown from (30) that $\Delta(e^2) \sim [g(q)]^{-1}$. In order to see this, we write (30) in the form

$$\Delta(e^2) = f[g(q(e^2))]/q(e^2) = f(g(x))/x = f(y)/h(y),$$

$$x = q(e^2), \qquad y = g(x),$$

and compute $f(y)/h(y)$ for large values of y:

$$\frac{f(y)}{h(y)} = f\left(\ln\frac{p^2}{m^2} + g'\right)\Big/\int_{\sim m^2}^{\infty} p^2\, d\varkappa^2 f\left(\ln\frac{\varkappa^2}{m^2} + g'\right)\Big/\varkappa^2(p^2 + \varkappa^2),$$

$$y = \ln(p^2/m^2). \quad (32)$$

Since we are concerned with the case of infinite renormalization, the values $\kappa^2 \sim p^2$ are important in the integral of formula (30), so that we may pull out of the integral the slowly varying function f at the point $\ln(p^2/m^2) + g'$. Then

$$\Delta(e^2) = f(y)/h(y) \approx \frac{1}{\ln(p^2/m^2)} = 1/y = 1/g(q(e^2)). \quad (33)$$

The function $\sigma(\kappa^2, e^2)$ may in this case go to zero for increasing e^2, if $d_c(1, e^2)\Delta(e^2) \to 0$, or it may stay finite and even increase. To resolve this question, we must understand the behavior of $d_c(1, e^2)$ for large values of the charge, which cannot be done in the framework of our techniques. It may turn out that, if $d_c(1, e^2)\Delta(e^2) \to 0$, that part of $\sigma(\kappa^2, e^2)$ becomes important which corresponds to the terms neglected in (4), as these vanish for $m^2/k^2 \to 0$. Nothing is known about the dependence of these terms on the charge. It should be noted, however, that by virtue of formula (9) the imaginary part of these terms for $k^2 < 0$, which is also the imaginary part of the total σ, is a negative quantity and can, therefore, never be greater (in terms of absolute values) than the calculated part of σ. In general, the case for which $\sigma(\kappa^2, e^2) \to 0$ is physically very improbable. Indeed, the function $\sigma(\kappa^2, e^2)$ is a sum of the squares of the moduli of the electromagnetic-field-operator matrix elements which

connect such states as, for example, the vacuum and an electron-positron pair. The condition $\sigma(\kappa^2, e^2) \to 0$ will imply, in particular, that as the charge increases, the cross section for the scattering of an electron by another electron with arbitrary momentum transfer much larger than m^2 decreases. If, on the other hand, the momentum transfer goes to zero, the quantum effects should disappear and the cross section should be of the Rutherford type, i.e., it should increase rapidly as a function of the charge! But this means that the cross section must change very rapidly in a relatively small interval of momentum transfers near $k^2 \sim m^2$.

All this seems, of course, very strange; however, one could argue that it is not unnatural to assume that electrodynamics has limited applicability for large charges. In any case, the idea that the theory becomes gradually worse as the charge increases seems rather phantastic.

There are thus two possibilities left, and the choice between these two is apparently very difficult. In the case $\nu > 0$ we have for d_c the functions (22) and for $\sigma(\kappa^2, e^2)$ formula (24), and in the case $\nu = 0$ formulas (29) and (31), respectively. These formulas have the character of expansions in terms of quantities which are small for strong coupling.

3. CONCLUSION

Our whole analysis of the behavior of the D function hinges on the possibility of making the renormalizability of the theory consistent with Lehmann's spectral representation (i.e., with the analyticity properties of the theory). By themselves these two suppositions are indeed not contradictory; however, this does not mean that the theory is internally consistent if it is examined fully, i.e., if we consider the Hamiltonian. For a better understanding of this circumstance we briefly outline the arguments of Landau and Pomeranchuk[1] concerning the "zero charge" in the light of our previous discussion.

As is known,[2] the connection between the renormalized and the bare charges e^2 and e_0^2, respectively, is (in our notation)

$$e_0^2 = h[g(q(e^2)) + L], \tag{34}$$

where $L = \ln(\Lambda^2/m^2)$ and Λ is the cut-off momentum. In going over to a local theory $\Lambda \to \infty$, and e_0^2 increases from small values until it reaches the value $\alpha = h(0)$ [the root of the function $g(x)$, see Figs. 1 and 2] for $L = -g[q(e^2)]$. In the neighborhood of this point the unrenormalized Green's function is equal to

$$e_0^2 d_0(\Lambda^2/k^2, e_0^2) = h[-(L-\xi) + g(e_0^2)] \approx h[-(L-\xi)] \tag{35}$$

and is, therefore, independent of e_0^2.

It is important that even if h has exponential character ($\nu \neq 0$), this assertion remains true, since $g(e_0^2)$ is small in comparison with unity for $e_0^2 \sim \alpha$. On the other hand, $e_0^2 d_0$ is the vacuum expectation value of the T product of two operators $U = e_0 A$ (A is the electromagnetic potential), and the fact that $e_0^2 d_0$ is independent of e_0^2 seems to be due to the neglect of the free term in the Lagrangian. (The latter is proportional to $e_0^{-2} U^+ U$, whereas the interaction is of the order of $U^+ U$.) Furthermore, in order to avoid a "zero charge", the function $g(e_0^2)$ should have the behavior shown in Fig. 2, which implies that $e_0^2 d_0$ will again start to depend on e_0^2. This seems surprising, not so far as the Lehmann expansion or the renormalizability property is concerned, but looking at it within the framework of the Lagrangian formalism; for the neglect of a term $\sim e_0^{-2} U^+ U$ in comparison with a term $\sim U^+ U$ can hardly become less justifiable as e_0^2 increases. There is thus good reason to assume that the functions g and h do not have the behavior shown in Figs. 1 and 2, if they are determined with the help of the Hamiltonian. Nevertheless, regardless of whether the zero charge does or does not exist in reality, it seems useful to study the general features of the theory. We summarize the results obtained:

1) In the real electrodynamics with $e^2 = 1/137$ at energies much larger than the critical value $[(e^2/3)\ln(k^2/m^2) \gg 1]$ the function d_c is either independent of the charge or exponentially small [formula (20)]. The last case corresponds to a non-logarithmic divergence of the exact expressions.

2) In an electrodynamics with a very large coupling constant it is possible (with an accuracy within a numerical constant) to determine the dependence of the function d_c on the energy in a fairly wide region. It has either the form (22) [the corresponding function σ is given by (24)] or the form (29) [the function σ is given by (31)].

In conclusion I should like to thank V. N. Gribov, K. A. Ter-Martirosyan, I. T. Dyatlov, and V. M. Shekhter for numerous useful comments.

[1] W. Pauli, Niels Bohr and the Development of Physics, London, Pergamon Press (1958).

[2] M. Gell-Mann and F. Low, Phys. Rev. **95**, 1300 (1954).

[3] Landau, Abrikosov, and Khalatnikov, Dokl. Akad. Nauk SSSR **95**, 497, 773, and 1177 (1954).

[4] H. Lehmann, Nuovo cimento **11**, 342 (1954).

[5] N. N. Bogolyubov and D. V. Shirkov Введение в квантовую теорию полей (Introduction to the Theory of Quantized Fields), Gostekhizdat (1957); Engl. Transl., Interscience, N.Y. (1959).

Translated by R. Lipperheide
244