A STUDY OF THE LIMITS OF APPLICABILITY OF THE THEORY OF IONIZATION LOSSES

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The energy losses of an arbitrary moving particle are calculated by means of the macroscopic Maxwell equations. A separation into ionization losses and radiation losses is made, outside the framework of perturbation theory. Effects on the formulas for ionization losses owing to multiple Coulomb scattering are examined, and also effects of the finiteness of the path length. It is found that because of the existence of the density effect the influence of multiple Coulomb scattering on this part of the losses can be neglected.

1. INTRODUCTION

IN this paper we study the limits of applicability of the theory of energy losses by excitation, ionization, and Cerenkov radiation at ultrahigh energies (hereafter for brevity we shall call these losses ionization losses). As will be shown below, for ionization losses, just as for radiation losses,[1] large longitudinal distances which increase with the energy of the particle are effective in a collision with an individual atom. This can be intuitively understood from the fact that the "collision times" $\Delta t_1 \approx (\rho/v)(1-v^2/c^2)^{1/2}$ (ρ is the impact parameter) correspond to large "passage times," during which the field of the particle is formed. From the relation

$$t = t_1 - R(t)/c = t_1 - \sqrt{v^2t^2 + \rho^2}/c$$

we at once get the required connection

$$t_{\text{eff}} \approx \rho/c\sqrt{1 - v^2/c^2} \quad \text{for} \quad t_1 = 0, \ \Delta t_{\text{eff}} \approx \Delta t_1/(1 - v/c).$$

Thus the field of the particle at the atom ($t_1 = 0$) during the time Δt_1 is determined by large passage-time intervals Δt_{eff} at large distances vt_{eff} from the atom.

If the effective distances become sufficiently large, various external causes can disturb the path of the charge, and this leads to a change in the theory of ionization losses. We shall consider the influences of three effects: multiple scattering, polarization of the medium, and finiteness of the trajectory. Inclusion of these effects in the radiation losses[2-4] has led to a sharp change of the formulas for bremsstrahlung and pair production. As regards the inclusion of the polarization of the medium in the calculation of ionization losses, its effect reduces to the well known density effect of Fermi and the Cerenkov radiation. Therefore the main part of this paper (Sec. 2 and 3) will be devoted to a study of the influence of multiple elastic scattering on the ionization losses.

For a qualitative estimate of this effect we consider the field produced by an arbitrarily moving charge. The Fourier components of the potentials are easily found from the Maxwell equations (cf. Sec. 2). For example, the vector potential is given by the expression

$$\mathbf{A}(\mathbf{k}, \omega) = -\frac{e}{4\pi^3 c}\left(\varepsilon\frac{\omega^2}{c^2} - k^2\right)^{-1}\int e^{-i\mathbf{kr}(t)+i\omega t}\,\mathbf{v}(t)\,dt,$$
$$\mathbf{A}(\mathbf{r}, t) = \int \mathbf{A}(\mathbf{k}, \omega)\,e^{i\mathbf{kr}-i\omega t}\,d\mathbf{k}\,d\omega.$$

If, starting from these expressions, we now calculate the flux of pseudoquanta, multiply it by the absorption coefficient of photons and integrate over $d\mathbf{k}$ and $d\omega$, we get an expression for the energy loss of an arbitrarily moving particle. The effect on the losses of deviations from rectilinear motion is determined by an integral of the form

$$\iint e^{-i\mathbf{k}(\mathbf{r}_1-\mathbf{r}_2)+i\omega(t_1-t_2)}\,dt_1\,dt_2,$$

in estimating which it is convenient to integrate over the direction of the vector $\mathbf{k}$.

If we forgo the scattering, then $|\mathbf{r}_1 - \mathbf{r}_2| = v(t_1 - t_2)$, $\omega = k_z v$, and the effective times will be of the order $t_{\text{eff}} \sim (\omega \pm kv)^{-1}$. Since $k = (\kappa^2 + \omega^2/v^2)^{1/2}$, for $\kappa \ll \omega/v$ we have

$$t_{\text{eff}} \sim \omega/\kappa^2 v \tag{1}$$

(we are using $t \sim (\omega - kv)^{-1}$, omitting the terms with $\omega + kv$).

Let us now estimate the effect of the scattering. In this case[2,3]

$$|\mathbf{r}_1 - \mathbf{r}_2| = v(t_1 - t_2) + \Delta r, \quad \Delta r = v^2 t^2 E_s^2/E^2 L$$

and in the exponent there is the additional quantity

$$k\Delta r \approx kv^2t^2E_s^2/E^2L \approx \omega vt^2E_s^2/2E^2L \tag{2}$$

(E is the energy of the electron, E_s is a constant of the order of 21 Mev, and L is the shower length). This quantity becomes comparable with the main term in the exponent, $(kv - \omega)t$, for a time t_1 that can be determined from the relation

$$(kv - \omega)\, t_1 \approx \omega v t_1^2 E_s^2/2E^2L. \tag{3}$$

Consequently, if $t_1 < t_{\text{eff}}$, then for momentum transfers in the direction perpendicular to the motion that satisfy

$$\varkappa_1^2 \lesssim (\omega E_s/Ev_1)\sqrt{\omega/Lv} \tag{4}$$

the scattering must be taken into account.

It is well known that there is a logarithmic contribution to the ionization loss from all values of the impact parameter (a quantity inversely proportional to κ) from minimum values to maximum values

$$\rho_{max} \approx \varkappa_{min}^{-1} \lesssim v/\omega_{\text{at}}\sqrt{1 - v^2/c^2} \tag{5}$$

(ω_{at} is a quantity of the order of atomic frequencies).

Comparing Eqs. (4) and (5), we see that the influence of scattering begins to be appreciable for the calculation of ionization loss if

$$\varkappa_1 \gtrsim \varkappa_{min},$$

that is, beginning at energies

$$E_1/mc^2 \gtrsim (L\omega_{\text{at}}/v)^{1/2}\, mc^2/E_s. \tag{6}$$

The situation is much changed, however, if we take into account the effect of the density of the medium. It will become clear from what follows that the density effect cuts off the effective range of κ at a value

$$\varkappa \approx \sqrt{4\pi Ne^2Z/mc^2} \tag{7}$$

(N is the density of particles in the medium). Comparing with Eq. (5), we find that the density effect becomes appreciable for

$$E_2/mc^2 \gtrsim (\omega_{\text{at}}/v)\sqrt{mc^2/4\pi NZe^2}, \tag{8}$$

that is, at smaller energies, and therefore the contribution to the ionization loss from the impact parameters at which the multiple scattering has a large effect is practically unimportant.

In the present paper we also consider the effect of the finiteness of the path on the ionization loss. Since lengths of the order of

$$ct_{\text{eff}} \sim c/(\omega - kv) \approx 2\omega_{\text{at}}/v\varkappa^2 \lesssim 2c/\omega_{\text{at}}(1 - v^2/c^2), \tag{9}$$

are effective for the ionization, the ionization curve must be changed if the length cT of the path is comparable with ct_{eff}. This condition sets in when

$$cT \lesssim (E/mc^2)^2\, c/\omega_{\text{at}}. \tag{10}$$

If the density effect is important at a given energy, we must substitute in Eq. (9) the value of κ from Eq. (7). Then we get

$$ct_{\text{eff}} \lesssim mc\omega_{\text{at}}/2\pi NZe^2 \tag{11}$$

In this case the finiteness of the path length will affect the ionization loss if

$$cT \lesssim mc\omega_{\text{at}}/2\pi NZe^2. \tag{12}$$

The effect of the finiteness of the path on the Cerenkov radiation was first considered by Tamm.[5]

2. THE TOTAL ENERGY LOSS

We shall calculate the energy loss by using the macroscopic Maxwell equations ($\mu = 1$):

$$\nabla^2 \mathbf{A} - \frac{\varepsilon}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} e\mathbf{v}\delta(\mathbf{r} - \mathbf{r}(t)),$$
$$\nabla^2 \varphi - \frac{\varepsilon}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -\frac{4\pi}{\varepsilon} e\delta(\mathbf{r} - \mathbf{r}(t)). \tag{13}$$

The solution for the potentials and the Lorentz condition are

$$\mathbf{A}(\mathbf{k},\omega) = \frac{e}{4\pi^3 c}\left[k^2 - \frac{\omega^2}{c^2}\varepsilon(\omega)\right]^{-1}\int_{-\infty}^{+\infty} \mathbf{v}(t)\, e^{i\omega t - i\mathbf{k}\mathbf{r}(t)}\, dt,$$
$$\varphi(\mathbf{k},\omega) = \frac{e}{4\pi^3\varepsilon(\omega)}\left[k^2 - \frac{\omega^2}{c^2}\varepsilon(\omega)\right]^{-1}\int_{-\infty}^{+\infty} e^{i\omega t - i\mathbf{k}\mathbf{r}(t)}\, dt,$$
$$\int_{-\infty}^{+\infty} (\mathbf{k}\mathbf{v}(t) - \omega)\, e^{i\omega t - i\mathbf{k}\mathbf{r}(t)}\, dt = 0. \tag{14}$$

The energy loss is given by the work done against the retarding force acting on the particle, according to the formula

$$F = -e\int_{-\infty}^{+\infty} \mathbf{E}(\mathbf{r},t)\, \mathbf{v}(t)\, dt, \tag{15}$$

where $\mathbf{E}(\mathbf{r}, t)$ is the electric field produced by the particle moving along the path $\mathbf{r}(t)$ with the velocity $\mathbf{v}(t)$. The Fourier component of the field is given by the relation

$$\mathbf{E}(\mathbf{k},\omega) = (i\omega/c)\,\mathbf{A}(\mathbf{k},\omega) - i\mathbf{k}\varphi(\mathbf{k},\omega). \tag{16}$$

Using these expressions, we easily get the formula that is the basis of our further calculations:

$$F = -\frac{ie^2}{4\pi^3}\int_{-\infty}^{+\infty} \frac{d\mathbf{k}\,\omega d\omega}{k^2 - \omega^2\varepsilon(\omega)/c^2}\left\{\left|\int \frac{\mathbf{v}}{c}\, e^{i\mathbf{k}\mathbf{r}(t) - i\omega t}\, dt\right|^2 - \frac{1}{\varepsilon}\left|\int e^{i\mathbf{k}\mathbf{r}(t) - i\omega t}\, dt\right|^2\right\}. \tag{17}$$

Regarding the limits of applicability of this formula we must make the following remarks. The energy $h\omega$ transferred to the medium must be much smaller than the energy of the electron, since $\mathbf{r}(t)$ is a prescribed trajectory, and recoil cannot be taken into account by such a method. In particular, since, as we shall see later on, Eq. (17) contains the loss to bremsstrahlung, the energy of the emitted quanta must be much smaller than that of the electron. For $\mathbf{r}(t)$ we must substitute the classical trajectory determined by the scattering by the Coulomb centers of force. Since, however, $Ze^2/hc < 1$, the concept of a classical trajectory is not valid, and the classical connection between the impact parameter and the angle of deviation does not exist. Actually this does not affect the result, because the procedure for obtaining the final result reduces to an averaging of the expression (17) over all possible trajectories, i.e., over all possible deviations from a straight path. Naturally, the averaging depends on the laws of multiple scattering, which are the same for the classical cases in virtue of the accidental circumstance that the Rutherford formula is always valid. Thus on this point the quantum nature of the phenomena is unimportant. Finally, we are using macroscopic electrodynamics. This means that the effective lengths must exceed interatomic distances, or in terms of momentum transfers this means that $\kappa \ll h/R_{at}$, where R_{at} is of the order of interatomic distances, and κ is the component of the vector $\mathbf{k}$ perpendicular to the direction of motion. This restriction must be made if we do not introduce explicitly the dependence of ϵ on the vector $\mathbf{k}$ (spatial dispersion), i.e., if we use the usual expression $\epsilon(\omega)$ which is legitimate in macroscopic electrodynamics.

For large values of κ we come to collisions with individual particles, so that a quantum calculation is necessary and we have to join it on to the formulas valid for small κ in a suitable way. On this point our calculations do not differ from the usual ones.

Finally, we shall make one more remark regarding the passage of Eq. (17) into an analogous formula when there is only one atom and we can neglect the effect of the medium. Here it is helpful to think of the loss as a quantity proportional to the imaginary part of $\epsilon(\omega)$, i.e., to the absorption coefficient. To get this result we replace the integration over ω between infinite limits by an integration from zero to infinity; using the property[6] of the dielectric constant, $\epsilon(-\omega) = \epsilon^*(\omega)$, we get

$$F_1 = -\frac{ie^2}{4\pi^3}\int\limits_{-\infty}^{+\infty} d\mathbf{k}\int\limits_0^{\infty}\frac{d\omega\,(\varepsilon-\varepsilon^*)\,\omega}{|k^2-\omega^2\varepsilon/c^2|^2}\left\{\left|\int\frac{\mathbf{v}}{c}e^{i\mathbf{kr}(t)-i\omega t}\,dt\right|^2\frac{\omega^2}{c^2}+\frac{k^2-\omega^2(\varepsilon+\varepsilon^*)/c^2}{|\varepsilon|^2}\left|\int e^{i\mathbf{kr}(t)-i\omega t}\,dt\right|^2\right\}. \quad (18)$$

The absorption coefficient is defined by the formula

$$N\sigma(\omega) = i\omega(\varepsilon-\varepsilon^*)/2c.$$

Here $\sigma(\omega)$ is the cross section for absorption of a quantum of frequency ω by an individual atom. Formula (18) can easily be obtained in a different way. We must calculate the flux of pseudoquanta from the moving particle, multiply it by the absorption coefficient, and integrate over all $\mathbf{k}$ and all ω. This makes clear how to make the passage to the case of collision with a single atom. To do this we must set $\epsilon = 1$ wherever it occurs in Eq. (18), except in the absorption coefficient. Then we are neglecting the influence of the density effect, and for a given motion $\mathbf{r}(t)$ we get as the formula for the loss:

$$F_2 = -\frac{ie^2}{4\pi^3}\int d\mathbf{k}\int\limits_0^{\infty}\frac{(\varepsilon-\varepsilon^*)\,\omega d\omega}{(k^2-\omega^2/c^2)^2}\left\{\left(k^2-\frac{2\omega^2}{c^2}\right)\left|\int e^{i\mathbf{kr}(t)-i\omega t}\,dt\right|^2 + \frac{\omega^2}{c^2}\left|\int\frac{\mathbf{v}(t)}{c}e^{i\mathbf{kr}(t)-i\omega t}\,dt\right|^2\right\}. \quad (19)$$

Let us now turn to the problem of separating out from the formula (17) the losses by ionization and bremsstralung. We first intergrate Eq. (17) over the direction of $\mathbf{k}$ relative to $\mathbf{r}(t_1) - \mathbf{r}(t_2)$. Then we get

$$F = -\frac{ie^2}{\pi^2}\int\frac{k^2dk\omega d\omega}{k^2-\omega^2\varepsilon(\omega)/c^2}\times\left[\iint\limits_{-\infty}^{+\infty}\left(\frac{\mathbf{v}_1\mathbf{v}_2}{c^2}-\frac{1}{\varepsilon}\right)e^{-i\omega(t_1-t_2)}\frac{\sin k|\mathbf{r}_1-\mathbf{r}_2|}{k|\mathbf{r}_1-\mathbf{r}_2|}dt_1\,dt_2\right]. \quad (20)$$

We shall assume the deviations from rectilinearity small and expand Eq. (20) in terms of these deviations. It must be noted that although the deviations are small the trigonometric functions can oscillate strongly, so that in general they cannot be expanded. If we denote the small deviations from rectilinearity by $\Delta\mathbf{r}$ and $\Delta\mathbf{v}$

$$\Delta\mathbf{r} = \mathbf{r}_2-\mathbf{r}_1-\mathbf{v}(t_2-v_1), \quad \Delta\mathbf{v} = \mathbf{v}_2-\mathbf{v}_1, \quad \mathbf{v} = \mathbf{v}_1 = \mathrm{const}$$

and introduce $t_1 - t_2 = t$ and t_1 as variables of integration, then since all the quantities in the integrand depend only on the difference of the times and the integral over t_1 gives the total time of passage, we shall have for the energy loss in unit time

$$F' = -\frac{ie^2}{\pi^2}\iint\frac{k^2dk\omega d\omega}{k^2-\omega^2\varepsilon(\omega)/c^2}\left\{\int\limits_{-\infty}^{+\infty}\left(\frac{v^2}{c^2}-\frac{1}{\varepsilon}+\frac{\mathbf{v}_1\Delta\mathbf{v}}{c^2}\right)\frac{\sin(kvt+k\Delta r)}{kvt+k\Delta r}\,dt\right\}. \quad (21)$$

We further average the expression (21) over all possible paths. We use the formulas of multiple scattering

$$\overline{\mathbf{v}\Delta\mathbf{v}} = -\theta^2 v^2/2 = -E_s^2 v^3|t|/2E^2L,$$

$$\overline{(\mathbf{r}_1-\mathbf{r}_2)^2} - \overline{v^2(t_1-t_2)^2} = -v^2 t\overline{\int_0^t \theta_\tau^2 d\tau} + v^2\overline{\left(\int_0^t \theta_\tau\, d\tau\right)^2}$$

$$= -v^2E_s^2t^2\,|\,t\,|\,/\,6E^2L. \tag{22}$$

Substituting Eq. (22) in Eq. (21), we get ($F' = F_1 = F_2$)

$$F_1 = -\frac{2ie^2}{\pi^2 v}\iint\frac{kdk\omega d\omega}{k^2-\omega^2\varepsilon(\omega)/c^2}\left(\frac{v^2}{c^2}-\frac{1}{\varepsilon}\right)$$

$$\times\int_0^\infty \cos\omega\, t\cdot\sin\left(kvt-\frac{kv^2t^2E_s^2}{12E^2L}\right)\frac{dt}{t},$$

$$F_2 = -\frac{2ie^2}{\pi^2 v}\iint\frac{kdk\omega d\omega}{k^2-\omega^2\varepsilon(\omega)/c^2}\left[\left(\frac{v^2}{c^2}-\frac{1}{\varepsilon}\right)\frac{vE_s^2}{12E^2L}-\frac{v^3E_s^2}{2c^2E^2L}\right]$$

$$\times\int_0^\infty \cos\omega t\,\sin\left(kvt-\frac{kv^2t^2E_s^2}{12E^2L}\right)dt. \tag{23}$$

The lack of exactness of the expressions is due to the use of the simple averaging procedure used by Landau and Pomeranchuk,[2] which reduces to the replacement of averages of products of functions by products of functions of the averaged arguments. For the first term, F_1, the lack of exactness is due only to the fact that the average value of the sine is replaced by the sine of the average value. In what follows we shall be interested in only the first term of the expansion. If in the argument of the sine we neglect the term proportional to E_s^2 we arrive at the usual expression for the ionization loss in distant collisions of a particle with uniform rectilinear motion (cf. Sec. 3).

For relativistic particles and for $\hbar\omega \gg \hbar\omega_{at}$ the second term in the expansion as given by Eq. (23) differs from the formula for bremsstrahlung[2,3] with inclusion of effects of multiple scattering and ionization of the medium only in its coefficient. This can be shown in the following way. For large energy transfers $\hbar\omega \gg \hbar\omega_{at}$, the absorption coefficient falls off sharply. This means that the medium becomes transparent, and in the integration over ω we can assume that ϵ has an infintely small imaginary part (we emphasize that for this approximation it is necessary that the thickness of the target be smaller than the length in which a quantum is absorbed). We now carry out the integration over the variable k in the expression (23), using the relation[6]

$$\int_{-\infty}^{+\infty} d\omega\int_0^{k_{max}}\frac{\omega k^2 dk A(\omega, k)}{k^2-(\omega^2/c^2)\,\varepsilon_1(\omega)-(i\omega^2/c^2)\,\varepsilon_2(\omega)}$$

$$=\int_0^\infty\frac{\pi i\omega^2}{c}\sqrt{\varepsilon}A\left(\omega,\ \frac{\omega}{c}\sqrt{\varepsilon}\right)d\omega. \tag{24}$$

In the derivation of this equation we have used the fact that ϵ_1 is an even function and $\epsilon_2 \to 0$ is an odd function of ω. The function $A(\omega, k)$ is an arbitrary even function of its arguments. Substituting the result (24) in the expression for F_2, we finally arrive at the formula for the bremsstrahlung

$$F_2 = -\frac{E_s^2}{E^2}\frac{e^2}{\pi L}\int_0^\infty d\omega\int_0^\infty\cos\omega t\sin\left(kvt-\frac{kv^2t^2E_s^2}{12E^2L}\right)dt, \tag{25}$$

where $k = \omega\epsilon^{1/2}/c$.

The formula (25) differs by a factor 3 from the analogous formula obtained in references 2 and 3 by the same method of calculation. The starting expression used in those papers differs from Eq. (17) by the presence of a factor $\mathbf{k}\cdot\mathbf{v}/\omega$ in the integrand of the second term in the curly brackets in Eq. (17). Using the Lorentz condition (15), however, one can easily verify the equivalence of the two expressions (in this case it is a matter of the emission of quanta larger than atomic energies, so that one must use Eq. (24) and the procedure of integration described above). The reason for the lack of agreement of the coefficients is that in references 2 and 3 terms were dropped which after an integration by parts make contributions to the integral (17) that are of the same order as the main terms.

Finally, let us examine the limits of applicability of the expansion (23), and also the connection between the distant and close collisions. As we have already pointed out, the expansion is in terms of the deviations from rectilinear motion, which are always small (we note that in such an expansion the degree of the terms in the integrand with respect to t increases, but the integration over t always extends effectively to t_{eff}, so that the expansion remains valid). Nevertheless we know that the radiation energy loss of a relativistic electron is larger than the loss by ionization. This is so owing to the presence of the factor $v^2/c^2 - 1/\epsilon$ in F_1 [Eq. (23)], i.e., an accidental circumstance which sharply decreases the loss at frequencies higher than atomic frequencies. Thus when the energy loss occurs in large portions the main contribution to the loss in the expression (23) is given by the second term. This is easy to show. In fact, the second term gives the energy loss in the emission of quanta of energies smaller than $\hbar\omega$ (the effect of the medium does not change the estimate):

$$F_2 \approx 4vr_0^2Z^2NL_{rad}\hbar\omega\,/\,137,$$

whereas the first term (we integrate it over all energy transfers, noting that the main contribution comes from atomic energies) gives

$$F_1 \approx 2\pi v Z e^4 N L_{ion} / mv^2.$$

A crude comparison of these two expressions shows that for large losses $\hbar\omega$ the second term is in fact more important than the first, and it would seem that to get a knowledge of F_1 we would need to use an exact method of averaging, so as to confirm that the small terms dropped from F_2 do not change the expression for F_1. The situation is much better than this, however, because hereafter we shall use the expression F_1 only for small energy transfers, of the order of atomic energies. For large energy transfers in the ionization loss we must use the formulas for free collisions. According to the calculations of Bethe,[7] already at energy transfers of the order of $5me^4/\hbar^2$ the main contribution to the ionization loss in hydrogen comes from free collisions. It is clear that for such values of $\hbar\omega$ the effect of F_2 is quite unimportant.

Thus in the range of energy transfers in which we are interested our approximation consists of the replacement of the average value of the sine by the sine of the average value, which leads to logarithmic accuracy, as we shall see below.

3. INVESTIGATION OF F_1

Let us rewrite F_1 in the form

$$F_1 = -\frac{2ie^2}{\pi^2 v}\int_0^\infty d\omega \int_0^\infty dk \frac{\omega k(\varepsilon - \varepsilon^*)}{|k^2 - \omega^2\varepsilon/c^2|^2 |\varepsilon|^2}\left[\frac{\omega^2}{v^2}\left|\frac{v^2}{c^2}\varepsilon - 1\right|^2 - \frac{\omega^2}{v^2} + k^2\right]\int_0^\infty \sin\left(kvt - \frac{kv^2t^2}{12L}\frac{E_s^2}{E^2}\right)\cos\omega t\,\frac{dt}{t} \tag{26}$$

and consider the integral over t:

$$I = I_1 + I_2 = \frac{1}{2}\int_0^\infty \sin\left[(\omega + kv)t - \frac{kv^2t^2}{12L}\frac{E_s^2}{E^2}\right]\frac{dt}{t} + \frac{1}{2}\int_0^\infty \sin\left[(kv-\omega)t - \frac{kv^2t^2E_s^2}{12LE^2}\right]\frac{dt}{t}. \tag{27}$$

In the first integral I_1 the effective range of integration is $t \sim 1/(\omega + kv)$. It is easy to show that the second term in the argument of the sine is always smaller than unity for all values of the variable of integration that are important for the integral I_1. That is, in I_1 we can always neglect the term with the scattering and integrate over t. We then find that $I_1 = \pi/4$ for all positive values of $\omega + kv$.

Let us now consider I_2. In the range

$$-(vE_s/E)\sqrt{k/12L} \gg kv - \omega \gg (vE_s/E)\sqrt{k/12L} \tag{28}$$

of the variables the term with the scattering can be neglected, and we have $I_2 = \pi/4$ if $\omega < kv$, and $I_2 = -\pi/4$ if $\omega > kv$. Since it can be seen from the condition (28) that the scattering is important only when $|kv - \omega|/kv \ll 1$, we can assume that $\omega \approx kv$. Outside the range (28), on the other hand, we can neglect the first term and get

$$I_2 = -\frac{1}{2}\int_0^\infty \frac{\sin\xi^2}{\xi}d\xi = -\frac{1}{4}\int_0^\infty \frac{\sin x}{x}dx = -\frac{\pi}{8}. \tag{29}$$

Let us rewrite F_1 in the following form:

$$F_1 = -\frac{2ie^2}{\pi^2 v}\int_0^\infty d\omega\left\{\frac{\pi}{8}\int_{\sim\varkappa_1}^{\sim\varkappa_2}\frac{\omega\varkappa(\varepsilon-\varepsilon^*)\,d\varkappa}{|k^2-\omega^2\varepsilon^2/c^2|^2|\varepsilon|^2}\left[\frac{\omega^2}{v^2}\left|\frac{v^2}{c^2}\varepsilon - 1\right|^2 + \varkappa^2\right] + \frac{\pi}{2}\int_{\sim\varkappa_2}^{\varkappa_{max}}\frac{d\varkappa\,\omega\varkappa(\varepsilon-\varepsilon^*)}{|k^2-\omega^2\varepsilon/c^2|^2|\varepsilon|^2}\left[\frac{\omega^2}{v^2}\left|\frac{v^2}{c^2}\varepsilon - 1\right|^2 + \varkappa^2\right]\right\}. \tag{30}$$

Here we have introduced instead of k the variable $\kappa^2 = k^2 - \omega^2/v^2$, which in the case of rectilinear motion is a quantity inversely proportional to the square of the impact parameter. In calculating the integral (30) we neglect the density effect. As we already know, to do this we must introduce the absorption coefficient, according to Eq. (19), and set $\epsilon = 1$ in the other parts of Eq. (30). After integrating over κ we get

$$F_1 = -\frac{Ne^2c}{\pi v}\int_0^\infty d\omega\sigma(\omega)\left\{\frac{1}{4}\ln\left/\frac{\varkappa_2^2 + \omega^2(1 - v^2/c^2)/v^2}{\varkappa_1^2 + \omega^2(1-v^2/c^2)/v^2}\right. + \ln\frac{\varkappa_{max}^2}{\varkappa_2^2 + \omega^2(1-v^2/c^2)/v^2} - \frac{v^2}{c^2} + \frac{\omega^2(1-v^2/c^2)/c^2}{\varkappa_2^2 + \omega^2(1-v^2/c^2)/v^2} - \frac{\omega^2(1-v^2/c^2)/c^2}{\varkappa_1^2+\omega^2(1-v^2/c^2)/v^2}\right\}. \tag{31}$$

To reduce the integral to final form we use the well known properties of the absorption coefficient:[6]

$$\int_0^\infty \omega\varepsilon''(\omega)\,d\omega = 2\pi^2 Ne^2 Z/m, \tag{32}$$

$$\ln\bar\omega = \frac{m}{2\pi^2 Ne^2 Z}\int_0^\infty \omega\varepsilon''(\omega)\ln\omega\, d\omega. \tag{33}$$

At first we neglect the term associated with the scattering. This is legitimate if the inequality opposite to the inequality (6) is satisfied. The formula (31) takes the form

$$F_1 = \frac{2\pi Ne^4 Z}{mv}\left\{\ln\frac{v^2\varkappa_{max}^2}{\bar\omega^2(1 - v^2/c^2)} - \frac{v^2}{c^2}\right\}. \tag{34}$$

If the inequality (6) holds, the formula for the loss takes a different form:

$$F_1 = \frac{2\pi Ne^4 Z}{mv}\left\{\ln\frac{\varkappa_{max}^2 E(12L)^{1/2}}{2\alpha(\bar\omega/v)^{3/2}E_s} - \frac{v^2}{c^2}\right\}, \quad \alpha \approx 1. \tag{35}$$

We shall now show that this case never occurs. The reason for this is the density effect, which we have temporarily neglected. If, on the other hand, we neglect the effect of the scattering, the total loss (in this case including also the loss by Cerenkov radiation) will be given by the formula of Fermi

$$F_1 = \frac{2\pi NZe^4}{mv}\ln\frac{mc^2\varkappa_{max}^2}{4\pi NZc^2}. \tag{36}$$

Comparing Eqs. (34) and (36), we note that the density effect begins to be appreciable when the condition (8) holds. A comparison of the conditions (6) and (8) shows that the density effect is always important at lower energies ($E_2 \ll E_1$). This means that when the energy is further increased (for $E > E_2$) the contribution to the ionization comes from the same impact parameters as for $E < E_2 \ll E_1$, at which the scattering still has no effect.

We can also examine the question of the influence of the scattering on the large-quantum energy loss. If $h \gg h\omega_{at}$, we can use the limiting value for the dielectric constant

$$\varepsilon = 1 - 4\pi NZe^2/m\omega^2. \tag{37}$$

It is easy to see that the factor $4\pi NZe^2/mc^2$ appears in the argument of the logarithm in Eq. (31). As can be verified without difficulty, the frequency range in which the scattering is of importance is given by

$$\left(\frac{4\pi NZe^2}{mc^2}\right)^{1/3}\left(\frac{E}{mc^2}\right)^{2/3}\frac{vL^{1/3}mc^2}{E_s} \ll \omega \ll \left(\frac{E_s}{mc^2}\right)^2\frac{vE^2}{Lm^2c^4} \ll \frac{E}{h} \tag{38}$$

and occurs at energies

$$E/mc^2 > (4\pi NZe^2/m)^{1/2}(mc^2/E_s)^2 L/c. \tag{39}$$

A comparison of Eqs. (38) and (39) with the analogous conditions in the formulas for the bremsstrahlung[3] shows that they are just the same. Furthermore, if the condition (38) is satisfied a pole appears in the integration of Eq. (30), and passage around this pole also leads to an additional contribution to the loss.

As we already noted above, however, these results do not change our conclusions about the total energy loss, because for large energy transfers the absorption coefficient is so small that the contribution to the loss calculated from Eq. (30) (without integration over ω) can be neglected in comparison with the contribution from free collisions (for energies that satisfy Eq. (39), Eq. (38) shows that the lower limit of the frequencies considerably exceeds atomic frequencies). Furthermore, the pole that appears in the first integral of Eq. (30) leads to a contribution to the emission of hard quanta satisfying the condition (38); but the intensity of the radiation is much smaller than that given by the term F_2. The latter term has been calculated inexactly because of the approximate method of averaging. Thus also for large energy transfers, although Eq. (30) is indeed changed, the contribution of F_1 to the large energy transfers can be neglected in comparison with the other losses.

4. INFLUENCE ON THE LOSSES OF THE EFFECT OF FINITENESS OF THE PATH

Let us consider the energy loss of a particle that has suddenly started its motion and has stopped after a time interval T. We shall neglect the effect of scattering. Integration of the basic formula (17) with the use of the Lorentz condition (15) gives

$$F = -\frac{2ie^2}{\pi^2}\int_0^{\varkappa_{max}}\iint_{-\infty}^{+\infty}\frac{\omega d\omega\varkappa\, d\varkappa\, dk_z}{\varkappa^2 + k_z^2 - \omega^2\varepsilon(\omega)/c^2}\left\{\frac{v^2}{c^2} - \frac{k_z^2v^2}{\varepsilon\omega^2}\right\}$$
$$\times\frac{\sin^2((k_z\vartheta - \omega)T/2)}{(k_zv - \omega)^2}. \tag{40}$$

To evaluate the integral we shall use a method described above. We at first neglect the density effect and look to see that change in the loss is caused by the presence of the limits on the trajectory. For this purpose we rewrite Eq. (40) in the form

$$F = -\frac{2ie^2}{\pi^2}\int_0^{\varkappa_{max}}d\varkappa\int_0^{\infty}d\omega\int_{-\infty}^{+\infty}dk_z\frac{\omega(\varepsilon-\varepsilon^*)\varkappa}{|\varepsilon|^2|\varkappa^2 + k_z^2 - \omega^2\varepsilon/c^2|^2}\left\{\left[\frac{\omega^2}{v^2}\left|\frac{v^2}{c^2}\varepsilon - 1\right|^2\right.\right.$$
$$\left.\left. - \frac{\omega^2}{c^2} + k^2\right]\frac{k_z^2v^2}{\omega^2} + \frac{|\varepsilon|^2v^2\omega^2}{c^4}\left(1 - \frac{k_z^2v^2}{\omega^2}\right)\right\}\frac{\sin^2((k_zv - \omega)T/2)}{(k_zv - \omega)^2}. \tag{41}$$

As we already know, the neglect of the density effect means that in Eq. (41) we are to set $\epsilon = 1$ everywhere in Eq. (41) except in the absorption coefficient (19). Then we can integrate over the parameter κ (we discuss the treatment of the pole later) and introduce the new variable of integration $x = (k_zv - \omega)T/2$.

We see from the expression (41) that the main contribution to the integral comes from the region $x \lesssim 1$ and $\omega \approx \omega_{at}$. We get (for $v \approx c$, $|k_zv - \omega|/\omega \ll 1$)

$$F = -\frac{ie^2T}{2\pi^2v}\int_0^{\infty}\omega\, d\omega(\varepsilon - \varepsilon^*)$$
$$\times\int_{-\infty}^{+\infty}dx\frac{\sin^2 x}{x^2}\left\{\ln\frac{c^2\varkappa_{max}^2}{2\omega^2|2x/T\omega + 1 - v/c|} - 1\right\}. \tag{42}$$

From Eq. (42) we at once get the conditions for the effect of finiteness of the path to be appreciable.

If the condition (10) is satisfied, then the existence of the limits on the path has no effect, and integrating over x we arrive at the usual loss theory [Eq. (34)]. In the opposite case we can neglect the quantity $1 - v/c$ in comparison with $2x/T\omega$ in the argument of the logarithm and integrate over x:

$$F = -\frac{ie^2T}{2\pi v}\int_0^\infty \omega\, d\omega\,(\varepsilon - \varepsilon^*)\ln\frac{Tc^2x_{max}^2}{4\omega}. \qquad (43)$$

Using Eqs. (32) and (33), we get

$$F = (2\pi Ne^4Z/mv)\ln(Tc^2x_{max}^2/4\bar{\omega}). \qquad (44)$$

This formula is valid if the logarithm is large.

Now let us estimate the influence of the density effect. As we have already seen, the density effect sets in when the condition (8) is satisfied. In this case the formula for the loss is Eq. (36). The combination of conditions (7) and (9) leads us to the criterion (12) given in the Introduction, under which the formula (43) holds. In this case the losses enter a plateau, the beginning and height of which are now determined not by the density effect, but by the finiteness of the path. It must be noted that the presence of limits to the path leads to the appearance of additional losses, which for a transparent medium appear in the form of radiation "at the stop."[5]

To separate this from the total loss [cf. Eq. (40) or (41)] we have to pass around the poles in the integration over κ. The passage around the poles [cf. Eq. (24)] leads to a formula for the loss by radiation

$$F_{\mathrm{rad}} = \frac{2c^2}{\pi}\int_0^\infty \omega\, d\omega \int_{-\infty}^{+\infty} dk_z \left\{\frac{v^2}{c^2} - \frac{k_z^2v^2}{\varepsilon\omega^2}\right\}\frac{\sin^2((\omega - k_zv)T/2)}{(\omega - k_zv)^2}. \qquad (45)$$

Let us rewrite the last formula, introducing the new variable

$$x = \omega - k_zv = 1 - (v\sqrt{\varepsilon}/c)\cos\vartheta \qquad (46)$$

(ϑ is the angle the emitted quantum makes with the x axis):

$$F_{\mathrm{rad}} = \frac{2c^2}{\pi v}\int_0^\infty \omega\, d\omega \int_{\omega(1-v\sqrt{\varepsilon}/c)}^{\omega(1+v\sqrt{\varepsilon}/c)} dx\left[\left(\frac{v^2}{c^2} - \frac{1}{\varepsilon}\right)\frac{\sin^2(xT/2)}{x^2} + \frac{2\sin^2(xT/2)}{\varepsilon\omega x} - \frac{\sin^2(xT/2)}{\varepsilon\omega^2}\right]. \qquad (45')$$

Suppose that

$$v\sqrt{\varepsilon}/c > 1, \qquad (47)$$

$$x_{min}T \gg 1. \qquad (48)$$

Then we can use the equation

$$x^{-2}\sin^2(xT/2) = \tfrac{1}{2}\pi T\delta(x)$$

and get the loss by Cerenkov radiation

$$F_C = T\frac{c^2v}{c^2}\int_{v\sqrt{\varepsilon}/c>0} \omega\, d\omega\,(1 - c^2/v^2\varepsilon). \qquad (49)$$

If the condition (47) is not satisfied, but Eq. (48) is, then we can average the rapidly oscillating factor (i.e., replace $\sin^2(xT/2)$ by $1/2$). Then Eq. (45′) can be rewritten in the form

$$F_{\mathrm{stop}} = -\frac{v^2e^2\sqrt{\varepsilon}}{\pi c^3}\int_0^\infty d\omega\int_0^\pi \frac{\sin^2\vartheta\, d(\cos\vartheta)}{(1 - (v\sqrt{\varepsilon}/c)\cos\vartheta)^2}. \qquad (50)$$

For $\epsilon = 1$ this last expression becomes the well known formula (with a factor 2) for the stopping radiation. If the condition (48) is not satisfied there is a change of the formulas that have been given, because of the effect of finiteness of the path. Equation (49) is practically unchanged. Equation (50) (sic) will be decidely changed, since the main contribution to the integral comes from the region around x_{min}. Here there is a complete analogy with the formulas for bremsstrahlung, and therefore the effect of multiple scattering which we omitted in Eq. (40) will also be very important at high energies.[2]

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