THE BEHAVIOR OF FERMION SPIN IN ELASTIC SCATTERING

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The behavior of spin in the elastic scattering of longitudinally polarized fermions and its dependence on the character of the interaction are investigated. It is shown that in the ultra-relativistic case (or for fermions with zero rest mass) the angle between the spin and the momentum is unchanged in V and A interactions, but the spin flips in S, P, and T interactions.

1. INTRODUCTION

THE motion of a free fermion will be described by a Dirac wave function that takes spin orientation into account,[1,2]

$$\psi = L^{-3/2}\sum_s C_s b_s e^{-i\varepsilon Kt + i\mathbf{kr}}, \qquad (1)$$

where the quantity $\epsilon = \pm 1$ characterizes the sign of the energy, and $s = \pm 1$ is twice the projection of the spin in the direction of the motion.

We represent the spin matrix b_s in the form

$$b_s = \frac{1}{\sqrt{2}}\begin{pmatrix} A_1 B_1 \\ A_1 B_2 \\ A_2 B_1 \\ A_2 B_2 \end{pmatrix}. \qquad (2)$$

Here the energy K of the fermion is connected with its momentum $\mathbf{k}$ and rest mass k_0 by the well-known relation $K = \sqrt{k^2 + k_0^2}$. The rest of the quantities in (2) are given by

$$A_1 = \sqrt{1 + k_0/\varepsilon K}, \quad A_2 = s\varepsilon\sqrt{1 - k_0/\varepsilon K},$$

$$B_1 = s\cos\theta_s e^{-i\varphi/2} \quad B_2 = s\sin\theta_s e^{i\varphi/2}, \quad \theta_s = \theta/2 - \pi(1-s)/4,$$

where θ and φ are the spherical angles of the vector $\mathbf{k}$.

The polarization properties of the fermions will be characterized by the four-component polarization pseudovector[3]

$$\zeta_\mu = K\int \psi^+ \sigma_\mu \psi\, d^3x, \qquad (3)$$

where the matrix σ_μ of the spin pseudovector is equal to σ and $\sigma_t = \rho_1$. Substituting (1) in (3) for the components of the polarization pseudovector, we find the values

$$\zeta_3 = Ks_3 = s_0^{-1}K\,(C_1^+C_1 - C_{-1}^+C_{-1}) \qquad (4)$$

for the longitudinal component, directed along the momentum, and

$$\zeta_1 = Ks_1 = s_0^{-1}k_0\,(C_{-1}^+C_1 + C_1^+C_{-1}) = k_0\cos\delta\sqrt{1 - s_3^2},$$

$$\zeta_2 = Ks_2 = s_0^{-1}k_0 i\,(C_{-1}^+C_1 - C_1^+C_{-1}) = k_0\sin\delta\sqrt{1 - s_3^2} \qquad (4a)$$

for the transverse components. Here δ is the phase difference between the complex amplitudes C_1^+ and C_{-1}, and the quantity

$$s_0 = C_1^+C_1 + C_{-1}^+C_{-1} \qquad (5)$$

inversely proportional to the normalization coefficient. The time component of the polarization is connected with the longitudinal component by the relation $\zeta_t = (k/K)\,\zeta_3$.

From this it is evident that the polarization properties of free fermions will be determined by two quantities: by the angle δ, characterizing the direction of the spin pseudovector $\mathbf{s}$ in the plane perpendicular to the momentum, where

$$\cos\delta = \zeta_1 / \sqrt{\zeta_1^2 + \zeta_2^2}, \quad \sin\delta = \zeta_2 / \sqrt{\zeta_1^2 + \zeta_2^2}, \qquad (6)$$

and by the angle α between $\mathbf{s}$ and the direction of the momentum $\mathbf{k}$, where

$$\tan\alpha = \frac{\sqrt{\zeta_1^2 + \zeta_2^2}}{\zeta_3} = \frac{k_0}{K}\frac{\sqrt{1 - s_3^2}}{s_3}. \qquad (7)$$

In investigating the transverse component, Ascoli[4] omitted from an analogous formula the factor k_0/K, since he considered the direction of a 3-component unit spin matrix

$$s_0\left(\frac{K}{k_0}s_1, \frac{K}{k_0}s_2, s_3\right).$$

As can be seen from (4) and (5), the modulus of the vector $\mathbf{s}$ is

$$|s| = \sqrt{s_3^2 + k_0^2K^{-2}(1 - s_3^2)}, \qquad (8)$$

where $k_0^2/K^2 = 1 - \beta^2$, and $c\beta$ is the velocity of the particle.

2. CALCULATION OF THE MATRIX ELEMENTS INCLUDING POLARIZATION EFFECTS

In investigating the transition of an electron from one initial state ($\epsilon = 1$, s, **k**) to some other final state ($\epsilon' = 1$, s', **k'**) we have to calculate matrix elements of the form

$$\gamma'_{\mu\nu} = b'^{+}\gamma_{\mu\nu}b = \frac{1}{2}\bar{\rho}_{\mu}(s', s)\,\bar{\sigma}_{\nu}(s', s), \tag{9}$$

where

$$\bar{\rho}_{\mu}(s', s) = (A_1'^{+}A_2'^{+})\,\sigma'_{\mu}\begin{pmatrix}A_1\\A_2\end{pmatrix}, \tag{10}$$

$$\bar{\sigma}_{\nu}(s', s) = (B_1'^{+}B_2'^{+})\,\sigma'_{\nu}\begin{pmatrix}B_1\\B_2\end{pmatrix}, \tag{11}$$

and σ'_{μ}, where $\sigma'_4 = 1$, are the well known two-rowed Pauli matrices. These formulas represent a generalization of the formulas given in reference 2 for the case of not only longitudinal, but also transverse polarization.

Without loss of generality, we can let the initial momentum **k** be directed along the z axis ($\theta = \varphi = 0$) and the final momentum **k'** be localized in the plane xz ($\varphi' = 0$). Then in elastic particle scattering (K' = K) we have

$$\bar{\rho}_{4,3}(s', s) = (1 \pm ss') + \frac{k_0}{K}(1 \mp ss'), \tag{12}$$

$$\bar{\rho}_{2,1}(s', s) = \frac{k}{K}s(1 \mp ss')\begin{pmatrix}-i\\1\end{pmatrix}, \tag{13}$$

$$\bar{\sigma}_{4,n}(s', s) = \frac{1}{2}(1 + ss')\begin{pmatrix}\cos(\theta'/2)\\ s(n_x + isn_y)\sin(\theta'/2) + n_z\cos(\theta'/2)\end{pmatrix}$$

$$+\frac{1}{2}(1 - ss')\begin{pmatrix}-s\sin(\theta'/2)\\ (n_x + isn_y)\cos(\theta'/2) - n_z\sin(\theta'/2)\end{pmatrix}, \tag{14}$$

where **n** is some three-component unit vector and θ' is the angle between the momenta **k'** and **k**.

3. THE BEHAVIOR OF FERMION SPIN IN ELASTIC SCATTERING

The interaction of a fermion with a fixed center can be described by the formula

$$U = \gamma u(\mathbf{r}).$$

The matrices $\gamma = \gamma_{\mu\nu}$ determine the character of the interaction (V, A, S, T, P).

We have the following expression for the differential effective cross section, taking polarization effects into account:

$$d\sigma_{s_1' s'} = C'^{+}_{s_1} C'_{s'}\frac{K^2}{4\pi^2 c^2\hbar^2}|u_{\mathbf{k}'\mathbf{k}}|^2 d\Omega, \tag{15}$$

where

$$u_{\mathbf{k}'\mathbf{k}} = \int e^{i\mathbf{r}(\mathbf{k}-\mathbf{k}')}u(\mathbf{r})\,d^3x,$$

$$C'_{s'} = b'^{+}(s')\gamma b(s), \qquad C'^{+}_{s_1} = b^{+}(s)\gamma b(s_1'),$$

and in the calculation of the summed (over the spins) differential effective cross section we must put into formula (15)

$$C'^{+}_{s_1}C'_{s'} \to s_0', \tag{15a}$$

where s_0' is determined by equality (5).

Let us examine the scattering of longitudinally-polarized fermions. We suppose that before the scattering the spin vector **s** was directed exactly along the particle momentum **k**, that is, $C_{s=1} = 1$, $C_{s=-1} = 0$. We then find what angle the vector **s'** of the scattered fermion will make with the direction of the new momentum **k'**.

The amplitudes characterizing the spin state s' of the scattered particle will be respectively

$$C_1' = b'^{+}\gamma_{\mu\nu}b = \frac{1}{2}\bar{\rho}_{\mu}(1, 1)\,\bar{\sigma}_{\nu}(1, 1),$$
$$C_{-1}' = \frac{1}{2}\bar{\rho}_{\mu}(-1, 1)\,\bar{\sigma}_{\nu}(-1, 1). \tag{16}$$

It must first be noticed that the representation of the 16 independent Dirac matrices in the form of a product of the matrices ρ_μ and σ_ν (see references 1 and 2) and not in the form of the standard γ matrices is convenient in the sense that the behavior of the spin (especially in the ultrarelativistic case) depends on the ρ_μ matrices.

a) We consider first the interactions proportional to the matrices ρ_1 and ρ_2. They correspond to the time component of the axial vector interaction $A^T = \rho_1\sigma_4$, the spatial component of the vector interaction $V^S = \rho_1\sigma_n$ (describing, for example, the scattering of an electric charge by a magnetic field), the pseudoscalar interaction $P = \rho_2\sigma_4$, and the time (more precisely, the time-space) component of the tensor interaction $T^T = \rho_2\sigma_n$, describing, for example, the scattering of a magnetic moment by an electric field.

As is evident from (14), the interactions V^S and A^T, proportional to the matrix ρ_1, lead to a matrix element that contains the factor $(1 + ss')$. In this case, if the momentum vector **k** and the spin vector **s** are parallel (s = 1) before the scattering, then after the scattering the new vectors **k'** and **s'** are still parallel (s' = 1). For the interactions P and T^T, which are proportional to the matrix ρ_2, the matrix element contains the factor $(1 - ss')$. Therefore after the scattering the vectors **k'** and **s'** must be antiparallel (s = 1, s' = −1).

For the quantity s_0' [see (15a)], characterizing the dependence of the summed (over spins) effective cross section, we find, respectively, the following four values, which depend on the character of the interaction (A^T, P, V^S, T^T):

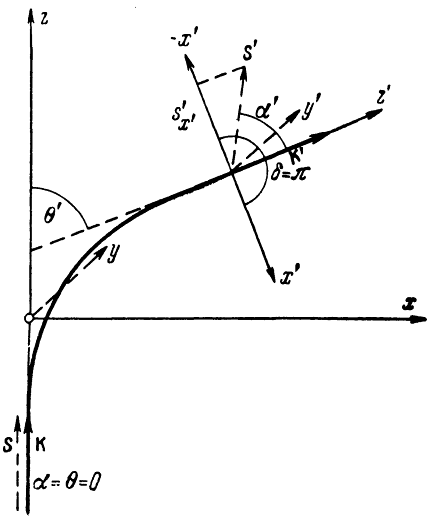
$$s_0' = \frac{1}{2}\frac{k^2}{K^2}\begin{pmatrix}1 \pm \cos\theta'\\ n^2 \pm N^2\end{pmatrix} \tag{17}$$

where

$$N^2 = 2\sin\theta' n_z n_x - \cos\theta'(n_x^2 + n_y^2 - n_z^2). \tag{18}$$

b) Let us now consider the interactions proportional to the matrices ρ_4 and ρ_3. To them belong the time component of the vector interaction $V^T = \rho_4\sigma_4$ (which describes, for example, the scattering of an electric charge by an electric field), the spatial component of the axial vector interaction $A^S = \rho_4\sigma_n$, the scalar interaction $S = \rho_3\sigma_4$, and the spatial component of the tensor interaction $T^T = \rho_3\sigma_n$ (which describes, for example, the scattering of a magnetic moment by a magnetic field).

In the case of the V^T and S interactions it is easy to show with the help of (4), (4a), and (6) that $\zeta_2' = 0$, $\cos\delta' = -1$, i.e., the spin will lie in the plane of the vectors **k** and **k′** and in the quadrants for which $x' < 0$ (see figure). For



the number s_0' and also for the angle α' between the spin **s′** and momentum **k′**, which is directed along the z' axis, we find correspondingly the values

$$s_0' = C_1'^{+}C_1' + C_{-1}'^{+}C_{-1}' = \tfrac{1}{2}\left[(1 + k_0^2/K^2) \pm \cos\theta'(1 - k_0^2/K^2)\right],$$

$$\tan\alpha' = 2\,\frac{k_0^2K^{-2}\sin\theta'}{(1 + k_0^2/K^2)\cos\theta' \pm (1 - k_0^2/K^2)}, \tag{19}$$

where the plus sign corresponds to the vector interaction and the minus sign to the scalar.

In the nonrelativistic case $k_0^2/K^2 = 1$ both interactions, as one might expect, lead to the unique result

$$s_0' = 1, \ \tan\alpha' = \tan\theta' \quad \text{or}\ \alpha' = \theta'. \tag{20}$$

From this, keeping in mind that the spin lies in the quadrants where $x' < 0$, we find, in particular, that for the scattering of an electric charge by an electric field the spin in the nonrelativistic case conserves its original direction (see reference 5), that is, in the specific case mentioned, the direction of the original momentum, $\mathbf{s}' \parallel \mathbf{k}$.

In the ultrarelativistic case $k_0^2K^2 \to 0$, with the help of (19) we find for the V^T interaction

$$\tan\alpha' = \frac{2k_0^2}{K^2}\tan\frac{\theta'}{2}. \tag{21}$$

For the S interaction we have

$$\tan\alpha' = -\frac{2k_0^2}{K^2}\cot\frac{\theta'}{2}. \tag{22}$$

From this it follows that as the energy increases the spin in the V^T interaction begins to turn around toward the direction of **k′** and coincides with this direction in the limiting case $k_0^2/K^2 = 0$.

In the S interaction, as the energy increases the spin begins to turn toward the direction opposite to that of the momentum, and in the limiting case $k_0^2/K^2 = 0$ the vectors **k′** and **s′** must be antiparallel ($s' = -1$).

For the A^S and T^S interactions we find

$$s_0' = \tfrac{1}{2}[n^2(1 + k_0^2/K^2) \pm (1 - k_0^2/K^2)N^2], \tag{23}$$

$$\tan\alpha' = \frac{2k_0^2}{K^2}\,\frac{\sqrt{n^4 - N^4}}{\pm n^2(1 - k_0^2/K^2) + (1 + k_0^2/K^2)N^2}, \tag{24}$$

where the plus sign corresponds to A^S and the minus sign to T^S and the number N^2 is determined by the formula (18).

In addition, in this case $\zeta_2' = (2k_0^2/s_0'K)n_zn_y \neq 0$ and therefore the polarization vector will form some angle with the (**kk′**) plane. In the ultrarelativistic case $k_0^2/K^2 = 0$ and therefore the angle α' will tend to zero ($s' = 1$) for the A^S interaction and to 180° for the T^S interaction ($s' = -1$).

In that way, we see that in the very relativistic case, when $k_0^2/K^2 \to 0$, only the V and A interactions, proportional to the matrices ρ_1 and ρ_4 conserve the parallelism of spin and momentum in the scattering process ($s' = s = 1$). In the case of S, T, and P interactions, proportional to ρ_2 and ρ_3, the spin after scattering changes its direction relative to the corresponding momentum to lie opposite ($s' = -s = -1$).

It is possible that this is connected with the fact that in the theory of the spin-oriented neutrino[2,6,7] where a spin reversal relative to the momentum is excluded because it would mean the transition of a neutrino to a nonexistent state, only the V and A variants are allowed.

4. MIXED INTERACTIONS

We want to apply the formulas obtained to the investigation of the scattering of fermions in the presence of a linear combination of interactions referring to different groups, where in the first group we have the interactions proportional to ρ_1 and ρ_4, and in the second those proportional to ρ_2 and ρ_3. We have such a combination of interactions, for example, in the analysis of the scattering of a particle which has an electric charge e and a "true" magnetic moment μ from a fixed point center having either an electric charge e′ or a magnetic moment μ'.

In this case the interaction energy has the form

$$U = e(\varphi - (\alpha \mathbf{A})) - \mu(\rho_3(\sigma \mathbf{H}) + \rho_2(\sigma \mathbf{E})), \tag{25}$$

where the field created by the point center is determined by the expression

$$\varphi = \frac{e'}{r}, \qquad \mathbf{E} = -\nabla \frac{e'}{r}, \qquad \mathbf{A} = \nabla \times \frac{\mu'}{r},$$

$$\mathbf{H} = \nabla \times \left[\nabla \times \frac{\mu'}{r}\right].^* \tag{26}$$

First we consider the behavior of the spin in the scattering of a particle with an electric charge ($e' \neq 0$, $\mu' = 0$).

Then for the matrix element entering in (15), we find, setting $s = 1$

$$C'_{s'=1} = ee' \cos\frac{\theta'}{2}, \tag{27}$$

$$C'_{s'=-1} = -e'\left(e\frac{k_0}{K} - 2\frac{k^2}{K}\mu\right)\sin\frac{\theta'}{2}, \tag{28}$$

from which we get an expression for the effective cross section

$$d\sigma = \frac{d\Omega e'^2}{4\sin^4(\theta'/2)\,k^4c^2\hbar^2}\left[e^2\left(K^2\cos^2\frac{\theta'}{2} + k_0^2\sin^2\frac{\theta'}{2}\right) + 4(k^4\mu^2 - k_0 e\mu k^2)\sin^2\frac{\theta'}{2}\right]. \tag{29}$$

In addition, it is easy to show that the spin vector will lie in the plane of the vectors **k** and **k′**, so that the tangent of the angle which the spin vector will form with the momentum **k′**, will be given by the expression

$$\tan\alpha' = \frac{k_0\,|(e^2k_0 - 2k^2\mu e)\sin\theta'|}{e^2\cos^2(\theta'/2)K^2 - \sin^2(\theta'/2)(ek_0 - 2k^2\mu)^2}. \tag{30}$$

It is evident from formulas (29) and (30) that in the nonrelativistic case $k \ll K \sim k_0$ a fundamental role will be played by the Coulomb interaction, i.e., V^T, and that therefore according to (20) the spin should preserve its direction, along the original momentum ($\alpha' = \theta'$). Then in the ultrarelativistic case $k_0 \ll K$, but with $\mu k \ll e$, the spin will turn aside toward the momentum **k′**. In the case of very large energies $\mu k \gg e$, when the dipole terms, proportional to ρ_2, play a fundamental role, the spin has to continue its turning until it has an orientation opposite to the final momentum **k′** ($s' = -1$).

Finally, we consider the spin behavior in the scattering of particles having an electric charge and a magnetic moment (e, μ) on a fixed magnetic moment ($\mu' \neq 0$, $e' = 0$). Here we distinguish two cases:

a) The magnetic moment μ' of the scattering center is parallel to the initial momentum ($\mu'_z = \mu'$, $\mu'_x = \mu'_y = 0$). Then

$$C'_1 = -(e + 2\mu k_0)\frac{2k^2\mu'}{K}\sin^2\frac{\theta'}{2}\cos\frac{\theta'}{2}, \quad C'_{-1} = 0, \tag{31}$$

and for the effective cross section we find the value

$$d\sigma = c^{-2}\hbar^{-2}\mu'^2(e + 2\mu k_0)^2\cos^2(\theta'/2)\,d\Omega', \tag{32}$$

i.e., in this case the scattering of a magnetic moment μ from a magnetic moment μ' will proceed just as the scattering of a charge $e_1 = 2\mu k_0$ from a magnetic moment μ' without the spin flip relative to the momentum **k′**. This is connected with the circumstance that the matrix element $\bar{\sigma}_n(-1, 1)$ describing the spin-flip scattering, which should give a basic dipole contribution to the effective cross section at ultrarelativistic energies $k \gg k_0$, goes to zero in the given case.

b) The magnetic moment μ' of the scattering center is perpendicular to the vector **k** ($\mu'_z = 0$, $\mu'^2_x + \mu'^2_y = \mu'^2 \neq 0$). Then

$$C_1 = -2\frac{k^2}{K}\sin\frac{\theta'}{2}\left[\mu'_x(e + 2\mu k_0)\sin^2\frac{\theta'}{2} + i\mu'_y\left(e + 2\mu k_0\sin^2\frac{\theta'}{2}\right)\right],$$

$$C'_{-1} = -4\mu\mu'_y\,ik^2\sin^2\frac{\theta'}{2}\cos\frac{\theta'}{2}. \tag{33}$$

From this we find the following values for the effective cross sections without spin flip ($s' = s = 1$):

$$d\sigma_{s'=1} = \frac{\mu'^2_x(e + 2\mu k_0)^2\sin^4(\theta'/2) + \mu'^2_y(e + 2\mu k_0\sin^2(\theta'/2))^2 d\Omega'}{c^2\hbar^2\sin^2(\theta'/2)} \tag{34}$$

and with spin flip ($s' = -1$)

$$d\sigma_{s'=-1} = 4(\mu/c\hbar)^2\mu'^2_y K^2\cos^2(\theta'/2)\,d\Omega'. \tag{35}$$

From formula (34) it is evident that in the case $2\mu k_0 \ll e$ the scattering without spin flip at arbitrary energies is basically due to the interaction of the charge e with the magnetic moment μ'.

The dipole terms should appear at higher energies, $\mu K \gg e$, when the scattering of a fermion takes place with spin flip.

Our final formulas may find application in in-

*If we consider a smearing of the scattering center, it is necessary to make a change which takes into account the contact term too[8]

$$\frac{1}{r} \to \frac{1}{r} - \frac{2\pi\overline{r_1^2}}{3}\delta(\mathbf{r})$$

where $\overline{r_1^2}$ is the mean square "smearing." This refinement will not be reflected in the investigation of the behavior of the spin in the scattering.

vestigating the scattering of polarized electrons, which along with the charge e must possess a vacuum, "true" magnetic moment, and also in the scattering of polarized protons or neutrons $(e = 0)$. It should be noted that experimental investigations of spin behavior in the scattering of polarized fermions may help determine the character of the interaction.

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26