EXCITATION OF VIBRATIONAL LEVELS IN NUCLEI BY CHARGED PARTICLES

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The excitation of the first vibrational level of an even-even nucleus by charged particles with energies close to the height of the Coulomb barrier is considered.

IN an earlier paper[1] the author considered the excitation of the rotational levels of nuclei by charged particles with energies close to the height, B, of the Coulomb barrier. The excitation of the vibrational levels can be discussed in an entirely analogous fashion. If the energy of the incoming particle is $E \sim B$, the basic process leading to collective excitations is, besides the ordinary Coulomb excitation, the direct nuclear interaction. Compound nucleus formation should play only a secondary role, since the compound nucleus can decay in a very great number of ways, out of which only one gives a contribution to the reaction under consideration. This is true especially in the case when the incoming particle is complex (α particle, deuteron, etc.).

Since the internal structure of the nucleus is not affected by the excitation of the collective levels, it is natural to describe the direct interaction with the help of the optical model, which is modified in such a way as to take into account the collective degrees of freedom in the nucleus. This modification consists of regarding the parameters specifying the shape of the complex nuclear potential as dynamical variables in the Schrödinger equation for the system consisting of the target nucleus and the incoming particle. Thus, if the nuclear potential has the form

$$V_n(\mathbf{r}) = V_n\left[\frac{r - R(\theta,\varphi)}{d}\right], \tag{1}$$

where

$$R = R_0\left[1 + \sum_{\lambda\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}(\theta,\ \varphi)\right], \tag{2}$$

one conveniently chooses for these collective coordinates the coefficients $\alpha_{\lambda\mu}$. The Hamiltonian of the system can then be written in the form

$$H = H_{vib}(\alpha) - (\hbar^2/2m)\nabla^2 + Z_1Z_2e^2/r + V_n(\mathbf{r}, \alpha) + V_c(r, \alpha), \tag{3}$$

where α is the set of coordinates $\alpha_{\lambda\mu}$, $\mathbf{r} = \mathbf{r}(r, \theta, \varphi)$ is the radius vector of the incoming particle, $V_c(\mathbf{r}, \alpha)$ is the non-central part of the electrostatic interaction leading to the Coulomb excitation, and $H_{vib}(\alpha)$ is the collective Hamiltonian of the nucleus:[2]

$$H_{vib}(\alpha) = \sum_{\lambda\mu}\left[\tfrac{1}{2}B_\lambda|\dot{\alpha}_{\lambda\mu}|^2 + \tfrac{1}{2}C_\lambda|\alpha_{\lambda\mu}|^2\right] \tag{4}$$

We are interested in a solution of the Schrödinger equation

$$H\Psi(\mathbf{r}, \alpha) = E\Psi(\mathbf{r}, \alpha), \tag{5}$$

which, for large r, has the form

$$\Psi(\mathbf{r}, \alpha) \to \varphi_{n_0}^{I_0\mu_0}(\alpha)\exp\{i\mathbf{k}_0\mathbf{r} + i\eta_{i0}\ln(kr - \mathbf{kr})\} + \sum_{I,\mu,n} r^{-1}f_{I\mu n}(\theta, \varphi)\varphi_n^{I\mu}(\alpha)\exp\{ik_nr - i\eta_{in}\ln k_nr\}. \tag{6}$$

Here $\varphi_n^{I\mu}(\alpha)$ are the wave functions of the stationary states of the nucleus, satisfying the equation

$$H_{vib}\varphi_n^{I\mu} = E_n\varphi_n^{I\mu}. \tag{7}$$

The oscillatory levels of the nucleus are characterized by the value of the total moment I, its projection μ, and the number of phonons n. The quantities k_n and η_n in (6) are the wave number and the Coulomb parameter $Z_1Z_2e^2/\hbar v$ of the incoming particle with energy $E - E_n$, respectively.

In the following we shall consider only the quadrupole ($\lambda = 2$) oscillations of even-even nuclei ($I_0 = \mu_0 = n_0 = 0$) with respect to the spherical equilibrium shape. Estimates show that the term $V_c(\mathbf{r}, \alpha)$ can here be regarded as a perturbation, as in the usual theory of the Coulomb excitation.

Our first task, therefore, is to solve the equation

$$H_0\Psi_0 = E\Psi_0, \tag{8}$$

where the Hamiltonian H_0 differs from the complete Hamiltonian (2) by the absence of the term $V_c(\mathbf{r}, \alpha)$.

We introduce the system of orthonormal functions

$$\Phi_{In}^{JM} = \sum_\mu C_{I\mu,\,IM-\mu}^{JM}\varphi_n^{I\mu}(\alpha)Y_{I,M-\mu}(\theta,\varphi); \tag{9}$$

these are the eigenfunctions of the total moment of

the system J, its projection M, the moment of the nucleus I, and the moment of the particle l.

The function Ψ_0 is conveniently expressed in a form analogous to that used in ordinary scattering theory:

$$r\Psi_0 = \sqrt{4\pi}\sum_l (kr)^{-1} i^{l+1}\sqrt{2l+1}\,\Psi_l^{(0)}(\mathbf{r},\alpha), \qquad (10)$$

where the functions $\Psi_l^{(0)}$ at $r \to \infty$ have the form

$$\Psi_l^{(0)} \to \Phi_{0l_0}^{l_0}\sin(k_0 r - l\pi/2 + \delta_l - \eta_0 \ln 2k_0 r) + \sum b_{0l_0,l'l'n'}^{(0)}\Phi_{l'l'n'}^{l_0}\exp\{ik_{n'}r - i\eta_{n'}\ln 2k_{n'}r\}. \qquad (11)$$

If the energy of the incoming particle is close to the height of the Coulomb barrier, the function $\Psi_l^{(0)}$ changes relatively slowly outside the nucleus for $r \sim R_0$, while it oscillates strongly inside the nucleus. This means that at the nuclear surface

$$\Psi_l^{(0)} \ll \partial\Psi_l^{(0)}/\partial(kr) \qquad (12)$$

so that we can make the approximation

$$\Psi_l^{(0)}[R(\theta,\varphi,\alpha)] = 0. \qquad (13)$$

The inequality (12) is not satisfied for certain resonance values of the energy; however, all these resonances lie above the Coulomb barrier and are therefore not important to us.

Outside the range of the nuclear forces the functions $\Psi_l^{(0)}$ can be expressed in the form

$$\Psi_l^{(0)} = F_l(k_0 r)\Phi_{0l_0}^{l_0} + \sum_{l'l'n'} b_{0l_0,l'l'n'}^{(0)}[G_{l'}(k_{n'}r) + iF_{l'}(k_{n'}r)]\Phi_{l'l'n'}^{l_0}, \qquad (14)$$

where $G_{l'}$ and $F_{l'}$ are the radial Coulomb functions, which for large r have the asymptotic forms

$$F_l \sim \sin(kr - l\pi/2 + \delta_l - \eta_l \ln 2kr),$$
$$G_l \sim \cos(kr - l\pi/2 + \delta_l - \eta_l \ln 2kr); \qquad (15)$$

the $b^{(0)}$ are certain unknown amplitudes. Condition (13) allows us to determine these amplitudes without considering the solution in the internal region at all. This condition can be rewritten in the form

$$F_l(k_0R)\Phi_{0l_0}^{l_0} + \sum_{l'l'n'} b_{0l_0,l'l'n'}^{(0)}[G_{l'}(k_{n'}R) + iF_{l'}(k_{n'}R)]\Phi_{l'l'n'}^{l_0}(\theta,\varphi,\alpha) = 0, \qquad (16)$$

where $R = R(\theta, \varphi, \alpha)$ is given by (2). Equation (16) must be satisfied for all values of θ, φ, α. The amplitudes $b^{(0)}$ are now found in exactly the same way as in the problem of the excitation of the rotational levels.[1]

By analogous methods we obtain

$$\beta_{0l_0,l'l'n'}^{(0)} = I_{0l_0,l'l'n'} - i\sum_{l'',l'',n''}\beta_{0l_0,l''l''n''}^{(0)} I_{l''l''n'',l'l'n'}F_{l''}(k_{n''}R_0)/G_{l''}(k_{n''}R_0), \qquad (17)$$

where

$$\beta_{0l_0,l'l'n'}^{(0)} = -b_{0l_0,l'l'n'}^{(0)}F_l(k_0R_0)/G_{l'}(k_{n'}R_0), \qquad (18)$$

$$I_{Iln,l'l'n'} = \int \Phi_{l'l'n'}^{l_0*}\frac{f_{Iln}}{\gamma_{l'l'n'}}\Phi_{Iln}^{l_0}\,d\Omega\,d\alpha, \qquad (19)$$

$$f_{Iln} = F_l[k_nR(\theta,\varphi,\alpha)]/F_l(k_nR_0);$$
$$\gamma_{Iln} = G_l[k_nR(\theta,\varphi,\alpha)]/G_l(k_nR_0). \qquad (20)$$

As in reference 1, we have neglected in (17) all integrals which differ from (19) in that f is replaced by γ. Since in all known cases the distance between vibrational levels is ~ 1 Mev, we have $F_l(k_nR_0)/G_l(k_nR_0) \ll 1$ for $E \lesssim B$ and $n > 0$. In the sum over n'' in (17) we may therefore keep only the terms with $n'' = 0$. In the following we shall be interested in the excitation of the first vibrational level ($n' = 0$, $I' = 2$); for the corresponding amplitude we have

$$b_{0l_0,2l'1}^{(0)} = -\frac{F_l(k_0R_0)}{G_{l'}(k_1R_0)}\frac{I_{0l_0,2l'1}}{1 + i[F_l(k_0R_0)/G_l(k_0R_0)]I_{0l_0,0l_0}} \qquad (21)$$

The integrals $I_{Iln,I'l'n'}$ can be calculated by expanding the logarithm of the function f/γ in powers of the quantity $\sum \alpha_{2\mu}Y_{2\mu}$, keeping only the linear term:

$$\frac{f_{ln}}{\gamma_{l'n'}} \to \exp\left\{k_0R_0\left[\frac{F_l'(k_0R_0)}{F_l(k_0R_0)} - \frac{k_1G_{l'}'(k_1R_0)}{k_0G_{l'}(k_1R_0)}\right]\sum_\mu \alpha_{2\mu}Y_{2\mu}\right\} \qquad (22)$$

(the prime denotes differentiation of the function with respect to its argument). Using the equations satisfied by the functions F_l and G_l and formulas (32) and (33), it is easily shown that the neglected terms have the order $1/\eta$. Substituting (22) in (19), we obtain

$$I_{0l_0,0l_0} = a_{l_0,l_0};\quad I_{0l_0,2l'1} = a_{l_0,l'1}\alpha_0\varkappa_{l_0,l'1}\sqrt{5/4\pi}C_{l_020}^{l'_0}; \qquad (23)$$

$$\varkappa_{ln,l'n'} = k_nR_0[F_l'(k_nR_0)/F_l(k_nR_0) - k_{n'}G_{l'}'(k_{n'}R_0)/k_{n'}G_{l'}(k_{n'}R_0)],$$
$$\alpha_0 = \hbar/\sqrt{2BC},\quad a_{ln,l'n'} = \exp\{(5\alpha_0^2/16\pi)\varkappa_{ln,l'n'}^2\}. \qquad (24)$$

Formulas (23), (24), and (20) solve the first part of the problem.

The exact amplitude b is connected with the amplitude found above, $b^{(0)}$, by the known relation

$$b_{Iln,I'l'n'} = b_{Iln,I'l'n'}^{(0)} + b_{Iln,I'l'n'}^{(c)}, \qquad (25)$$

$$b_{Iln,I'l'n'}^{(c)} = -\frac{2m}{k\hbar^2}\int \Psi_l^{(-)*}V_c\Psi_{l'}^{(0)}\,d\mathbf{r}\,d\alpha, \qquad (26)$$

where $\Psi_l^{(-)}$ is the l-th component in the expansion of the exact wave function $\Psi^{(-)}$, analogous to (10); for large r, $\Psi_l^{(-)}$ consists of an incident plane wave and an incoming spherical wave.

As already indicated, we regard V_c as a perturbation, and we can therefore replace the exact function $\Psi_l^{(-)}$ in (26) by $\Psi_l^{(-)(0)}$. It is easily seen

that we can also neglect the diverging wave in (26) which is due to the scattering by the nucleus. This leads to a small error in the total cross sections which is connected with the fact that the nuclear amplitudes $b^{(0)}$ decrease much faster with increasing l than the Coulomb amplitudes $b^{(C)}$.

Retaining in (26) only the "incoming" waves and assuming for simplicity that the charge density in the nucleus is constant, we find

$$b^{(c)}_{0l0,2l'1} = -(6\gamma/5)\, k_0 k_1 R_0^2 \alpha_0 \sqrt{5/4\pi}\, C^{l'0}_{l020}\, M^{-3}_{ll'}, \qquad (27)$$

where $M^{-3}_{ll'}$ is a radial matrix element defined by

$$M^{-3}_{ll'} = \frac{1}{k_0 k_1} \int_0^\infty F_{l'}(k_1 r)\, r^{-3} F_l(k_0 r)\, dr. \qquad (28)$$

For the total cross section for the excitation of the first vibrational level we now have

$$\sigma_{0\to 1} = \sigma^{(c)}_{0\to 1} + \sigma^{(n)}_{0\to 1} + \sigma^{(nc)}_{0\to 1},$$

where

$$\sigma^{(c)}_{0\to 1} = 4\pi k_0^{-2} \sum_{ll'} (2l+1)\, |b^{(c)}_{ll'}|^2 =$$

$$(45/16\pi^2) k_0^{-2} \eta_i^2 (k_1/k_0)\, R_0^2 \alpha_0^2 (R_0/a)^2 f_{E2}(\eta, \xi), \qquad (29)$$

$$\sigma^{(n)}_{0\to 1} = 4\pi k_0^{-2} \sum_{ll'} (2l+1) |b^{(0)}_{0l0,2l'1}|^2, \qquad (30)$$

$$\sigma^{(nc)}_{0\to 1} = 8\pi k_0^{-2} \sum_{ll'} (2l+1)\, \mathrm{Re}[b^{(c)}_{0l0,2l'1}\, b^{(0)*}_{0l0,2l'1}]. \qquad (31)$$

Here $a = Z_1 Z_2 e^2/2E$, $\xi = \eta_0 - \eta_1$, and $f_{E2}(\eta, \xi)$ is a dimensionless function known in the theory of Coulomb excitation (tables for this function can be found, for example, in the review article by Alder et al.[3]).

To calculate $\sigma^{(n)}$ and $\sigma^{(nc)}$ it is necessary to know the Coulomb functions $F_l(kr)$ and $G_l(kr)$ near the classical turning point $kr = 2\eta$. In this region these functions are given with good accuracy by the expressions[4]

$$F_l = (2\eta_l)^{1/6} v\left[\frac{\eta + \sqrt{\eta^2 + l(l+1)} - kr}{(2\eta)^{1/3}}\right], \qquad (32)$$

$$G_l = (2\eta_l)^{1/6} u\left[\frac{\eta + \sqrt{\eta^2 + l(l+1)} - kr}{(2\eta)^{1/3}}\right], \qquad (33)$$

where u and v are Airy functions in the Fock notation.[5] These functions are related to the Bessel functions of order $1/3$ in the following way:

$$\begin{matrix} u(t) \\ v(t) \end{matrix} = \sqrt{\frac{\pi}{3}\, t}\left\{I_{-1/3}\left(\frac{2}{3} t^{3/2}\right) \pm I_{1/3}\left(\frac{2}{3} t^{3/2}\right)\right\} \qquad t > 0,$$

$$\begin{matrix} u(t) \\ v(t) \end{matrix} = \sqrt{\frac{\pi}{3}\, t}\left\{J_{-1/3}\left(\frac{2}{3}|t|^{3/2}\right) \mp J_{1/3}\left(\frac{2}{3}|t|^{3/2}\right)\right\} \qquad t < 0$$

[the upper sign refers to the function u(t)].

The important values of l in the sums (30) and (31) are $l \sim (2\eta)^{2/3}$. If the incoming particle is a proton, a typical value of η is $\eta \sim 3$. In this case it is easy to carry out the summations in (30) and (31) immediately. To evaluate the radial matrix elements $M^{-3}_{ll'}$ in (31), it is convenient to use the approximate quasi-classical expressions for them, which are quite accurate for $\eta > 1$:

$$M^{-3}_{ll'} = I_{2\mu}(\theta, \xi)/4\eta_i^2, \qquad \mu = l - l',$$

$$\theta = 2\sin^{-1}[1 + l(l+1)\eta^{-2}]^{-1/2},$$

where $I_{2\mu}(\theta, \xi)$ is a classical orbital integral known in the theory of Coulomb excitation.[6] Tables of the function $I_{2\mu}(\theta, \xi)$ are given, for example, in reference 3.

If the excitation of the nucleus is due to α particles, then $\eta \gg 1$ ($\eta \sim 10$) and the summation in (30) and (31) can be carried out analytically (with an accuracy up to terms of order $1/\eta$). Let us first consider the sum (30). It is seen from the expression (21) for the amplitudes $b^{(0)}_{ll'}$ that for $\eta \gg 1$ only the Clebsch-Gordan coefficient $C^{l'0}_{l020}$ depends strongly on the value l' for a given l; in the other factors we may set $l' = l$.

Since the Airy function is quite sensitive to changes of its argument of order unity, the root in (32) and (33) may be expanded in terms of powers of $l(l+1)/\eta^2$, keeping only the linear term. Furthermore, we can replace the summation over l in (30) by an integration over the variable $x = l(l+1)/(2\eta)^{4/3}$. We then obtain for $\sigma^{(n)}$

$$\sigma^{(n)} = 5(2\eta_i)^{2/3} R_0^2 \alpha_0^2 \int_0^\infty \left\{ a^2(z_0, z_1, x)\left[\frac{k_1 u'(x+z_1)}{k_0 u(x+z_1)} - \frac{v'(x+z_0)}{v(x+z_0)}\right]^2 \right.$$

$$\left. \times \frac{u^2(x+z_0)\, v^2(x+z_0)}{u^2(x+z_1)[u^2(x+z_0) + a^2(z_0, z_0, x)\, v^2(z_0+x)]} \right\} dx,$$

where

$$z_n = (2\eta_n - k_n R_0)/(2\eta_n)^{1/3},$$

$$a(z_n, z_p, x) = \exp\left\{\frac{5\alpha_0^2}{16\pi}(k_n R_0)^2\left[\frac{k_p u'(z_p+x)}{k_n u(z_p+x)} - \frac{v'(z_n+x)}{v(z_n+x)}\right]^2\right\}$$

The integrand can now be written in the form

$$\left\{a^2(z_0, z_1, x)\frac{v(z_0+x)\, u^3(z_0+x)}{u^2(z_1+x)}\left[\frac{k_1 u'(z_1+x)}{k_0 u(z_1+x)} - \frac{v'(z_0+x)}{v(z_0+x)}\right]^2\right\}$$

$$\times \left\{\frac{v(z_0+x)}{u^3(z_0+x)[1 + a^2(z_0, z_0, x)\, v^2(z_0+x)/u^2(z_0+x)]}\right\}$$

For all values of the excitation energy of the first vibrational level occuring in real cases, the first factor depends very weakly on x as compared to the second factor (in a typical case, $\eta_0 = 10$; $k_0 R_0 = 2\eta_0$ and $\eta_1 = 9.5$, it varies by less than 20%, whereas the second factor changes by more than two orders of magnitude). This comes from the fact that in the considered region of values of the arguments, the Airy functions change monotonically, while the logarithmic derivatives u'/u and v'/v, the products v(x)u(x),

and the ratios $u(x+a)/u(x+b)$ vary slowly (not exponentially) in comparison with expressions of the type u^{-2} or v/u.

The first factor can therefore be pulled out from under the integral sign, with $x = 0$. To calculate the remaining integral it is convenient to change to a new variable of integration, $t = v(z_0+x)/u(z_0+x)$; by virtue of the relation $u'v - uv' = 1$ we have

$$dx / u^2(z_0 + x) = dt.$$

The slowly varying quantity $a(z_0, z_0, x)$ must be regarded as a constant, setting $x = 0$.

As a result we obtain

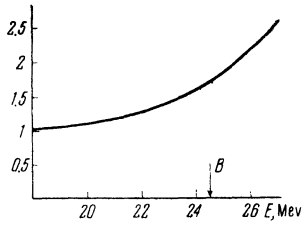
$$\sigma^{(n)} = {}^5/_2 R_0^2 (2\gamma_l)^{2/3} \alpha_0^2 \frac{a^2(z_0, z_1, 0)}{a^2(z_0, z_0, 0)} \frac{v(z_0) u^3(z_0)}{u(z_1)} \times \left[\frac{k_1 u'(z_1)}{k_0 u(z_1)} - \frac{v'(z_0)}{v(z_0)}\right]^2 \ln\left[1 + a^2(z_0, z_0, 0) \frac{v^2(z_0)}{u^2(z_0)}\right]. \quad (34)$$

The summation in expression (31) for $\sigma^{(nc)}$ can be carried out in an entirely analogous fashion. The factor $I_{2\mu}(\theta, \xi)$ can be taken out of the integral with $\theta = \pi$ (i.e., $l = 0$).

We finally obtain for $\sigma^{(nc)}$:

$$\sigma^{(nc)} = 6(k_1 R_0) R_0^2 \alpha_0^2 \left[\frac{k_1 u'(z_1)}{k_0 u(z_1)} - \frac{v'(z_0)}{v(z_0)}\right] \frac{a(z_0, z_1, 0)}{a(z_0, z_0, 0)} \times \frac{v(z_0) u^2(z_0)}{u(z_1)} I_{20}(\pi, \xi) \tan^{-1}\left\{a(z_0, z_0, 0) \frac{v(z_0)}{u(z_0)}\right\}. \quad (35)$$

We note that all quantities $a(z, z, 0)$ are very close to unity for the parameter values occuring in reality.



The figure shows the ratio of the total cross section for the excitation of the first vibrational level over the cross section for Coulomb excitation by α particles for the nucleus $_{52}Te^{120}$ ($E_1 = 0.56$ Mev).

In the case of protons the nuclear corrections are more important.

We note in conclusion that in the case of α particles we can replace the boundary condition (12) by the condition of complete absorption, accounting for the presence of a diffuse boundary in the same way as was done in the discussion of the excitation of the rotational levels. Our discussion is easily generalized to the case $I_0 \neq 0$, $\lambda \neq 2$. The excitation of higher vibrational levels ($n > 1$) has to be considered together with terms of higher orders in V_c. This is connected with great difficulties, since the adiabatic approximation is not applicable in this case.

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