*VELOCITY DISTRIBUTION OF ELECTRONS IN A STRONG ELECTRIC FIELD**

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Submitted to JETP editor June 12, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 1394-1400 (November, 1959)

A method is developed for finding a nonstationary solution of the Boltzmann equation in the case of strong electric fields. An expression is derived for the electron distribution function in a completely ionized plasma located in a strong electric field. It is shown that in the first approximation the distribution is a Maxwellian one superimposed on the general translational motion of the electron gas. In the first approximation the translational velocity increases proportionally with time, whereas the temperature remains constant.

IN the solution of the problem of the velocity distribution of electrons in an electric field usually the investigation is restricted to the stationary case. Moreover, the method of successive approximations is used, which is based on expanding the distribution function $f_e(\mathbf{v})$ in spherical harmonics in velocity space.[1,2] However, this method is justified only when the anisotropic part of the distribution function is much smaller than its isotropic part, or, which is the same thing, when the mean directed velocity is smaller than the thermal velocity. In the stationary case this condition, as a rule, is fulfilled, as a consequence of the small exchange of energy ϵ between the electrons and the gas atoms ($\Delta\epsilon/\epsilon \sim 2m/M \ll 1$). However, for times shorter than the relaxation time, the mean directed velocity may exceed the thermal velocity, and thus the above solution method turns out, generally speaking, to be inapplicable to the description of the processes leading to the establishment of a steady state. In a number of cases, however, such processes are of an essentially nonstationary character. Thus, for example, it can be shown that if the cross section for the collision of electrons with heavy particles falls off faster than $1/v$, then no stationary state exists at all in the presence of a constant electric field. The nonexistence of a stationary distribution is associated with the appearance of so-called "run-away" electrons.[3-5] The "run-away" phenomenon consists of the following. The energy lost by an electron per unit time is proportional to the collision frequency $\nu(v)$. Evidently if $\nu(v)$ falls off with the velocity faster than $1/v$, then when the electron velocity exceeds a certain critical value the increase in the electron energy under the action of the external electric field becomes larger than the energy losses due to collisions, and therefore, the electron energy will continuously increase with time.*

This effect is most pronounced in the case of fully ionized gases,† when the particles are subject to a Coulomb interaction and, consequently, the collision frequency is given by $\nu(v) \sim v^{-3}$. However, in the case of weak fields and not very high temperature, the number of such electrons is very small, although it does increase with time. Therefore for limited time intervals in a number of problems one can still utilize the stationary solution for the distribution function, if one formally cuts it off at large velocities (of the order of several times the thermal velocity). But if the electric field is sufficiently large, so that even over one mean free path the electrons acquire sufficient energy to enter the state of continuous acceleration, then the number of "run-away" electrons is large, the mean directed velocity may considerably exceed the thermal velocity, and the usual method of solution becomes inapplicable. In this case it is natural to utilize another method of obtaining an asymptotic solution, which consists in expanding the distribution function in inverse powers of the electric field, taking for the first approximation the solution of the Boltzmann equation without collisions.

As an illustration of the application of the method

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*Reported at the IV International Conference on Ionization Phenomena in Gases (Upsala, August 1959).

*An attempt at a theoretical investigation of this phenomenon on the assumption that the ratio of the ion mass to the electron mass is infinite ($m/M = 0$) was made by Dreicer,[4] who reported the results of a numerical integration of the Boltzmann equations for the case of a fully ionized plasma.

†Strictly speaking the "run-away" electrons are present at any degree of ionization, even when collisions with neutral atoms play the dominant role, since at sufficiently high velocities the screening becomes unimportant, and the electrons are, in fact, scattered by the atomic nuclei. However, in this case the number of such electrons is very small and, as a rule, they need not be considered.

outlined above we shall obtain the velocity distribution of electrons in a fully ionized homogeneous plasma situated in a strong electric field. Let us first consider the case when the electric field is constant in time.

The equation from which the electron distribution function $f_e(\mathbf{v}, t)$ may be determined has the following form

$$\frac{\partial f_e}{\partial t} + \gamma \frac{\partial f_e}{\partial \mathbf{v}} = \mathrm{St}\{f_e\},$$
$$\mathrm{St}\{f_e\} = \mathrm{St}_{ee}\{f_e f_e\} + \mathrm{St}_{ei}\{f_e f_i\}, \qquad (1)$$

where $\boldsymbol{\gamma} = -e\mathbf{E}/m$, the terms $\mathrm{St}_{ee}\{f_e f_e\}$ and $\mathrm{St}_{ei}\{f_e f_i\}$ take into account the electron-electron and electron-ion collisions, while $f_i(\mathbf{v})$ is the ion distribution function.

By following the procedure outlined above and by setting

$$f_e(\mathbf{v}, t) = f_e^{(1)}(\mathbf{v}, t) + f_e^{(2)}(\mathbf{v}, t) + \ldots, \qquad (2)$$

we obtain

$$\frac{\partial f_e^{(1)}}{\partial t} + \gamma \frac{\partial f_e^{(1)}}{\partial \mathbf{v}} = 0,$$
$$\frac{\partial f_e^{(n)}}{\partial t} + \gamma \frac{\partial f_e^{(n)}}{\partial \mathbf{v}} = \mathrm{St}\{f_e^{(n-1)}\}, \qquad n = 2, 3, \ldots \qquad (3)$$

From this equation, letting $f_0(\mathbf{v})$ denote the initial distribution function

$$f_0(\mathbf{v}) = f_e(\mathbf{v}, t)|_{t=0},$$

we obtain

$$f_e^{(1)}(\mathbf{v}, t) = f_0(\mathbf{v} - \gamma t),$$
$$f_e^{(n)}(\mathbf{v}, t) = \int_0^t dt' \int \mathrm{St}\{f_e^{(n-1)}(\mathbf{v}', t')\}\, \delta(\mathbf{v}' - \mathbf{v} - \gamma[t' - t])\, d\mathbf{v}'. \qquad (4)$$

We note that from the relation

$$\int \mathrm{St}\{f_e\}\, d\mathbf{v} = 0$$

it follows that

$$\int f_e^{(n)}(\mathbf{v}, t)\, d\mathbf{v} = 0 \qquad (n = 2, 3, 4, \ldots) \qquad (5)$$

and, thus, the normalizing factor is determined just by the first approximation to the function.

In order to calculate $f_e^{(n)}(\mathbf{v}, t)$ we must know the collision operator $\mathrm{St}\{f_e\}$. The expression for this operator is given by Landau.[6] It has the following form

$$\mathrm{St}_{ee}\{f_e f_e\} = -\alpha \frac{\partial}{\partial v_k} \int \left[f_e \frac{\partial f_e'}{\partial v_n'} - f_e' \frac{\partial f_e}{\partial v_n} \right] V_{kn}(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}', \qquad (6)$$

$$\mathrm{St}_{ei}\{f_e f_i\} = -\alpha \frac{\partial}{\partial v_k} \int \left[\delta f_e \frac{\partial f_i'}{\partial v_n'} - f_i' \frac{\partial f_e}{\partial v_n} \right] V_{kn}(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}', \qquad (7)$$

where summation over $n, k = 1, 2, 3$ is implied;

$$f_e' = f_e(\mathbf{v}', t), \qquad f_i' = f_i(\mathbf{v}', t), \qquad \delta = m/M,$$
$$V_{kn} = (u^2\delta_{kn} - u_k u_n)/u^3, \qquad u_k = v_k - v_k', \qquad \alpha = 2\pi e^4 \lambda / m^2,$$

while λ is a slowly varying function of the particle velocity. The expression for λ obtained by Landau is applicable only to a low temperature plasma ($e^2/\hbar v > 1$). In our case we must use a different expression given by Landshoff.[7] Since λ is insensitive to the choice of the values of the density and the velocity we will in future suppose that $\lambda = \mathrm{const}$.

Let us suppose for the sake of simplicity that the ions have a Maxwellian distribution corresponding to the temperature T_0, i.e.,*

$$f_i(\mathbf{v}) = N(\beta_i/2\pi)^{3/2} \exp(-\beta_i v^2/2),$$
$$f_0(\mathbf{v}) = N(\beta_e/2\pi)^{3/2} \exp(-\beta_e v^2/2), \qquad (8)$$

where $v^2 = v_1^2 + v_2^2 + v_3^2$, while $\beta_e = \delta\beta_i = m/kT_0$. We choose the z axis in the direction opposite to the electric field vector $\mathbf{E}$. Then it follows from (4) that

$$f_e^{(1)}(\mathbf{v}, t) = N(\beta_e/2\pi)^{3/2} \exp\left\{-\frac{1}{2}\beta_e[v_1^2 + v_2^2 + (v_3 - \gamma t)^2]\right\}, \qquad (9)$$

where $\gamma = |\boldsymbol{\gamma}|$.

We substitute expressions (8) and (9) into (7) and integrate over the velocities. Then, on taking into account the fact that $\mathrm{St}_{ee}\{f_e^{(1)} f_e^{(1)}\} = 0$, and neglecting quantities of order δ compared to unity, we obtain

$$\mathrm{St}\{f_e^{(1)}\} = -\alpha\beta_e\beta_i f_e^{(1)}(\mathbf{v}, t)\,[v_3 \Psi(v\sqrt{\beta_i/2}) - \delta\gamma t \chi(\mathbf{v})]\, v^{-1}\gamma t, \qquad (10)$$

where

$$\Psi(z) = z^{-2}(\Phi(z) - z\partial\Phi/\partial z), \qquad \Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2} dy,$$

$$\chi(\mathbf{v}) = (1 - v_3^2/v^2)\,[\Phi(v\sqrt{\beta_i/2}) - \tfrac{1}{2}\Psi(v\sqrt{\beta_i/2})] + (v_3^2/v^2)\,\Psi(v\sqrt{\beta_i/2}). \qquad (11)$$

Now on substituting the expression obtained for $\mathrm{St}\{f_e^{(1)}\}$ into (4), and on integrating, we obtain to the same order of accuracy after introducing the notation $v_r^2 = v_1^2 + v_2^2$, $v_z = v_3$:

*In general, the ion distribution function in strong fields need not have the form (8). However, in the case under consideration at present the specific form of $f_i(\mathbf{v})$ has only a small effect on the final result.

$$f_e^{(2)}(\mathbf{v}, t) = \frac{2\alpha N\beta_e}{\gamma} f_e^{(1)}(\mathbf{v}, t)\left\{\frac{\beta_e v_r^2 (v_z - \gamma t)\left[(v_z - \gamma t)^2 + v_r^2 + 2/\beta_i\right]}{2(v_r^2 + 2/\beta_i)(v^2 + 2/\beta_i)^{1/2}}\right.$$

$$+ \gamma t \frac{2/\beta_i + v_r^2 + {}^1/_2\beta_e v_r^2\left[(v_z - \gamma t)^2 - v_r^2\right]}{(v_r^2 + 2/\beta_i)(v^2 + 2/\beta_i)^{1/2}}$$

$$- \frac{\beta_e v_r^2 (v_z - \gamma t)\left[(v_z - \gamma t)^2 + v_r^2 + 2/\beta_i\right]^{1/2}}{2(v_r^2 + 2/\beta_i)}$$

$$\left. - \left(1 - \frac{\beta_e v_r^2}{2}\right)\ln\frac{(v^2 + 2/\beta_i)^{1/2} + v_z}{\left[(v_z - \gamma t)^2 + v_r^2 + 2/\beta_i\right]^{1/2} + (v_z - \gamma t)}\right\}.$$

By proceeding in a similar manner we can, in principle, determine $f_e(\mathbf{v}, t)$ to any predetermined degree of accuracy. However, since the successive approximations are of no fundamental interest, while the expressions obtained for them are rather awkward, we shall limit ourselves to the second approximation. In this case we have

$$f_e(\mathbf{u}, \tau) = N\left(\frac{\beta_e}{2\pi}\right)^{3/2} e^{-u^2}\left\{1 - \varepsilon\left[\frac{u_z u_r^2 (u^2 + \delta)^{1/2}}{u_r^2 + \delta}\right.\right.$$

$$- \frac{\varepsilon u_z u_r^2 (u^2 + \delta) + \tau\left[\delta + u_r^2 (1 + u_z^2 - u_r^2)\right]}{\varepsilon (u_r^2 + \delta)\left[(u_z + \tau/\varepsilon)^2 + u_r^2 + \delta\right]^{1/2}}$$

$$\left.\left. + (1 - u_r^2)\ln\frac{\left[(u_z + \tau/\varepsilon)^2 + u_r^2 + \delta\right]^{1/2} + u_z + \tau/\varepsilon}{(u^2 + \delta)^{1/2} + u_z}\right]\right\}, \qquad (12)$$

where the following dimensionless variables have been introduced

$$u^2 = {}^1/_2\beta_e v^2, \quad u_r^2 = {}^1/_2\beta_e v_r^2, \quad u_z^2 = {}^1/_2\beta_e (v_z - \gamma t)^2, \quad \tau = \nu_0 t,$$

$$\nu_0 = \sqrt{2}\alpha N \beta_e^{3/2}, \quad \varepsilon = \gamma_k/\gamma, \quad \gamma_k = eE_k/m = 2\alpha N\beta_e. \qquad (13)$$

By comparing the successive terms of the expansion (2) we can easily obtain a sufficiency criterion for the applicability of the method outlined above. It has the form

$$E \gg E_k \ln(4\tau E/\delta E_k),$$

where in accordance with the definition [cf. (13)]

$$E_k = 2 \cdot 10^{-12} N/T_0 \ \text{v/cm},$$

if the temperature T_0 is expressed in electron-volts. We note that if we are interested only in the calculation of averages, then it is sufficient to require that the series (2) should converge only for velocities which in order of magnitude are equal to the thermal velocity. In this case the condition for convergence is somewhat relaxed and assumes the following form

$$E \gg E_k \ln(2\tau E/E_k). \qquad (14)$$

By utilizing the expression (12) obtained above for $f_e(\mathbf{v}, t)$ we can determine the time dependence of the average directed velocity $\mathbf{z}(\tau) = (\frac{1}{2}\beta_e)^{1/2}\overline{\mathbf{v}}$, and of the square of the components of the mean squared relative velocity along and at right angles to the electric field: $w_{\parallel}(\tau) = \frac{1}{2}\beta_e(\overline{v_z^2} - \overline{v}_z^2)$ and $w_{\perp}(\tau) = \frac{1}{2}\beta_e\overline{v_r^2}$. However, it is simpler to proceed directly from equations (3). On multiplying each of them by $\mathbf{v}$, adding them, and integrating over the velocities, we obtain after neglecting, as was done previously, quantities of order δ compared to unity

$$\mathbf{z}(\tau) = (\gamma/\gamma_k)\left\{\tau - (\varepsilon^2/\sqrt{\pi})\left[1 - {}^1/_2\tau^{-1}\varepsilon\sqrt{\pi}\Phi(\tau/\varepsilon)\right]\right\}. \qquad (15)$$

Similarly we obtain

$$w_{\perp}(\tau) = 1 + \varepsilon\left\{\left[\Phi\left(\frac{\tau}{\varepsilon}\right) - \frac{3}{2}\Psi\left(\frac{\tau}{\varepsilon}\right)\right]\ln\left(\frac{\tau}{\varepsilon}\right)\right.$$

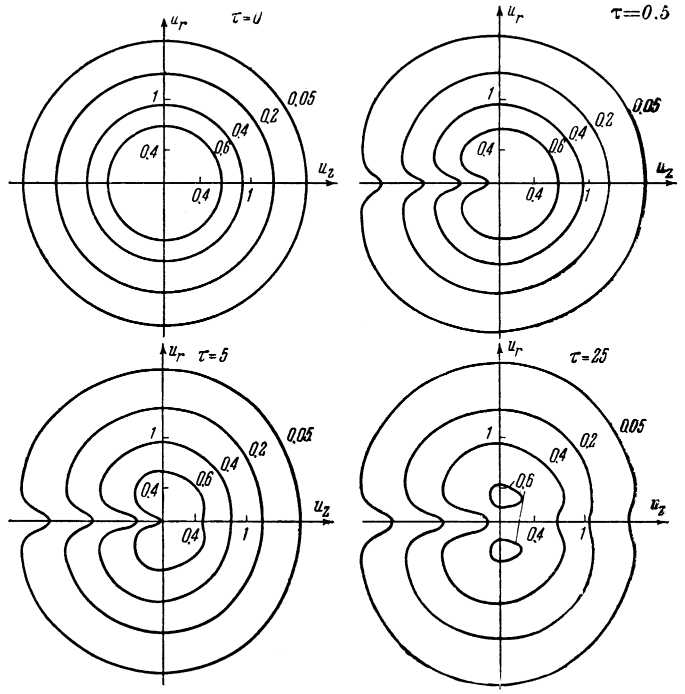
$$\left. - 3\int_0^{\tau/\varepsilon}\left[\frac{\Psi(x)}{x} - \frac{4}{3\sqrt{\pi}}e^{-x^2}\right]\ln x\,dx\right\}, \qquad (16)$$

$$w_{\parallel}(\tau) = \frac{1}{2} + \varepsilon\left\{\frac{3}{2}\int_0^{\tau/\varepsilon}\frac{\Psi(x)}{x}\,dx - \Phi\left(\frac{\tau}{\varepsilon}\right)\right.$$

$$\left. - \frac{\varepsilon}{\pi}\left[1 - \frac{\varepsilon\sqrt{\pi}\Phi(\tau/\varepsilon)}{2\tau}\right]^2\right\}, \qquad (17)$$

i.e., $w_{\perp}$ increases logarithmically with time, while $w_{\parallel}$ remains practically constant.

The figure shows the contours of the distribution function for different instants of time in terms of the dimensionless variables u_r, u_z for the case $\epsilon = 5 \times 10^{-2}$. It is interesting to note that in the region of small u_r the distribution function has a dip which increases with time, i.e., the number of particles with zero radial velocity continuously diminishes. It is also characteristic that the distribution function is asymmetric in the direction of the electric field. The physical reason for this asymmetry lies in the difference in the initial conditions for particles which have different directions of velocities at the instant when the field is switched on. Thus, for example, the velocity of those electrons, which have been moving against the field at the initial instant of time, will continually increase, and, consequently, the probability of their colliding with ions will decrease, while the velocity of electrons with negative values of v_z will at first decrease, and the probability of collision will increase. In general, the particles at the trailing edge of the distribution function ($u_z < 0$), undergo a larger number of collisions than the particles at the leading edge ($u_z > 0$), and this leads to the asymmetry indicated above. The fact that the distribution function has a "gap" is apparently also related to the previously mentioned dependence of the collision frequency ν on the velocity. Indeed, those electrons which have a low radial velocity have a greater probability of colliding than electrons with



Lines of constant values of the distribution function $N^{-1}[\beta_e/2\pi]^{-3/2} f_e(\mathbf{u}, \tau)$.

the same longitudinal velocity v_z, but with a larger value of v_r, as a result of which the number of particles with small values of v_r decreases continually.

Thus we see that in a strong electric field the electron velocity distribution stays close to the initial distribution (with the exception of a small range of u_r close to zero) superimposed on the average translational velocity $\mathbf{z}(\tau)$, and merely spreads slowly with time in the direction perpendicular to the electric field vector.

In the above discussion we assumed that the electric field $\mathbf{E}$ was constant in time. However, this assumption is not essential, and the method of solution outlined above may be easily generalized to the case when the electric field depends on the time. Indeed, on setting $\mathbf{E}(t) = \mathbf{E}\varphi(t)$, where $\mathbf{E}$ is constant, and on taking into account the fact that the replacement of the variable t by $x(t) = \int_0^t \varphi(\xi)\,d\xi$ brings the equation

$$\frac{\partial f_e}{\partial t} + \varphi(t)\,\gamma\frac{\partial f_e}{\partial \mathbf{v}} = \mathrm{St}\,\{f_e\} \tag{1'}$$

into the form (1), we evidently obtain in place of (4)'

$$f_e^{(1)}(\mathbf{v}, t) = f_0(\mathbf{v} - \gamma\,x(t)),$$

$$f_e^{(2)}(\mathbf{v}, t) = \int_0^t dt' \int \mathrm{St}\,\{f_e^{(1)}(\mathbf{v}', t')\}\,\delta(\mathbf{v}' - \mathbf{v} - \gamma\,[x(t') - x(t)])\,d\mathbf{v}'. \tag{4'}$$

If, moreover, the initial distribution is Maxwellian, we have

$$\mathrm{St}\,\{f_e^{(1)}\} = -\alpha\beta_e\beta_i f_e^{(1)}(\mathbf{v}, t) \times \frac{\gamma x(t)}{v}\left[v_3\Psi\left(v\sqrt{\frac{\beta_i}{2}}\right) - \delta\gamma x(t)\,\chi(\mathbf{v})\right] \tag{10'}$$

and the problem of finding the second approximation to the function reduces, as before, to a single integration. Finally, the expressions for the average directed velocity $\mathbf{z}(\tau)$, $w_\perp(\tau)$, and $w_\parallel(\tau)$ may be obtained in the same way as in the case $\varphi(t) = 1$. After a few simple transformations we find

$$\mathbf{z}(\tau) = \frac{\gamma}{\gamma_k}\left\{\nu_0 x(\tau/\nu_0) - \frac{\varepsilon\nu_0}{2}\int_0^{\tau/\nu_0} \Psi\left(\frac{\nu_0 x(\zeta)}{\varepsilon}\right) d\zeta\right\}, \tag{15'}$$

$$w_\perp(\tau) = 1 + \varepsilon\int_0^{\tau/\nu_0}\left[\Phi\left(\frac{\nu_0 x(\zeta)}{\varepsilon}\right) - \frac{3}{2}\Psi\left(\frac{\nu_0 x(\zeta)}{\varepsilon}\right)\right]\frac{d\zeta}{x(\zeta)}, \tag{16'}$$

$$w_\parallel(\tau) = \frac{1}{2} + \nu_0\frac{\gamma}{\gamma}\int_0^{\tau/\nu_0} \mathbf{z}(\nu_0\zeta)\,\Psi\left(\frac{\nu_0 x(\zeta)}{\varepsilon}\right) d\zeta - \varepsilon\int_0^{\tau/\nu_0}\left[\Phi\left(\frac{\nu_0 x(\zeta)}{\varepsilon}\right) - \frac{3}{2}\Psi\left(\frac{\nu_0 x(\zeta)}{\varepsilon}\right)\right]\frac{d\zeta}{x(\zeta)}, \tag{17'}$$

where the functions $\Phi(z)$ and $\Psi(z)$ are defined in accordance with (11). The criterion for the applicability of these formulas may be obtained from the condition that the correction terms in formulas (16'), (17') should be small compared to the principal terms. It has the following form

$$w_\perp(\tau) - 1 \ll 1. \tag{14'}$$

We note that in the case when the electric field increases monotonically with time, it follows from expressions (16') and (17') that the quantities $w_\perp(\tau)$ and $w_\parallel(\tau)$ remain bounded, i.e., in contrast to the case $\varphi(t) = 1$ the electron "temperature" tends to a certain constant value which is equal to

$$T_{max} = {}^2/_3 T_0\,[w_\perp(\infty) + w_\parallel(\infty)].$$

In conclusion we note that if we do not make the assumption that the initial electron temperature T_e is equal to the ion temperature T_0, then for $\delta T_0/T_e \ll 1$ the formulas obtained earlier for f_e, $\mathbf{z}$, $w_\perp$, and $w_\parallel$ remain valid if in them we replace T_0 by T_e, and δ by $\delta T_0/T_e$.

The author is indebted to M. S. Rabinovich for useful discussions.

Note added in proof (October 21, 1959). In this paper we have assumed that the ion distribution function is given. The simultaneous solution of equations for the electrons and the ions shows that the results referring to the electrons remain unaltered. Analogous expressions are obtained for the ions, with the expansion parameter ϵ remaining the same, i.e., the concept of a strong field is the same both in the cases of ions and electrons.

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Translated by G. Volkoff
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