TRANSITION RADIATION IN WAVEGUIDES

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We examine the radiation produced in a waveguide when a charged particle passes through the boundary between two media. It is shown that at ultrarelativistic charge velocities the radiation is mainly in the forward direction and its magnitude is proportional to the particle energy. Formulas are derived for the radiation energy and for its spectral distribution.

TRANSITION radiation[1-3] may in the near future find extensive application in fast-particle counters and in ultrashort-wave generators. The question of transition radiation in a waveguide is therefore of interest.

1. Let us consider an arbitrary cylindrical waveguide with perfectly conducting walls, filled with two homogeneous dielectrics, the constants of which are ϵ_1 and ϵ_2 for $z < 0$ and $z > 0$ respectively. A charged particle moves parallel to the axis of the waveguide from the negative z direction towards the separation boundary with velocity v. The field in the waveguide is described by a vector potential $\mathbf{A}(0, 0, A)$, the Fourier representation of which satisfies the equation

$$\nabla^2 A_{\omega i} + \frac{\varepsilon_i \omega^2}{c^2} A_{\omega i} = -\frac{4\pi}{c} j_\omega, \qquad i = 1, 2, \qquad (1.1)$$

where j_ω is the Fourier component of the particle current. The field vectors are obtained from the relations

$$\mathbf{E}_\omega = -\frac{i\omega}{c}\mathbf{A}_\omega + \frac{c}{i\omega\varepsilon}\operatorname{grad}\operatorname{div}\mathbf{A}_\omega, \qquad \mathbf{H}_\omega = \operatorname{curl}\mathbf{A}_\omega, \qquad (1.2)$$

with $A_\omega = 0$ on the surface of the waveguide.

The solution of (1.1) is represented in the form of a field connected with the particle, and a free field, which is a "flash" of transition radiation, occurring during the reshaping of the particle field at the instant when it passes from one medium to another. The first field has the form

$$A^0_{\omega 1} = \frac{2e}{c} e^{-i\omega z/v} \sum_{n=1}^{\infty} \frac{\psi_n(M_0)\,\psi_n(M)}{\lambda_n^2 + (\omega/v)^2(1-\beta^2\varepsilon_1)},$$

$$A^0_{\omega 2} = \frac{2e}{c} e^{-i\omega z/v} \sum_{n=1}^{\infty} \frac{\psi_n(M_0)\,\psi_n(M)}{\lambda_n^2 + (\omega/v)^2(1-\beta^2\varepsilon_2)}, \qquad (1.3)$$

where $\psi_n(M)$ and λ_n are the normalized eigenfunction and eigenvalue of the first boundary problem for the transverse section of the waveguide:

$$\nabla^2\psi_n + \lambda_n^2\psi_n = 0 \qquad (1.4)$$

with $\psi_n = 0$ on the surface of the waveguide, and M_0 is the point of intersection of the trajectory of the charge with the transverse section.

The free field is written in the form of a system of propagating waves

$$A'_{\omega 1} = \frac{2e}{c}\sum_{n=1}^{\infty} A_n\psi_n(M_0)\,\psi_n(M)\,e^{i\Gamma_n z},$$

$$A'_{\omega 2} = \frac{2e}{c}\sum_{n=1}^{\infty} B_n\psi_n(M_0)\,\psi_n(M)\,e^{-i\gamma_n z}, \qquad (1.5)$$

with

$$\Gamma_n = \sqrt{\omega^2\varepsilon_1/c^2 - \lambda_n^2}, \qquad \gamma_n = \sqrt{\omega^2\varepsilon_2/c^2 - \lambda_n^2},$$

$$\operatorname{Im}\Gamma_n < 0, \qquad \operatorname{Im}\gamma_n < 0,$$

for $\omega > 0$, and a complex-conjugate expression for $\omega < 0$. The unknown coefficients are found from the condition of continuity of the tangential components of the vectors at $z = 0$:

$$A_n = \frac{1}{\varepsilon_1\Gamma_n + \varepsilon_2\gamma_n}\left[\frac{\varepsilon_2\omega/v - \varepsilon_1\gamma_n}{\lambda_n^2 + (\omega/v)^2(1-\beta^2\varepsilon_1)} - \frac{\varepsilon_1\omega/v - \varepsilon_1\gamma_n}{\lambda_n^2 + (\omega/v)^2(1-\beta^2\varepsilon_2)}\right],$$

$$B_n = \frac{1}{\varepsilon_1\Gamma_n + \varepsilon_2\gamma_n}\left[\frac{\varepsilon_2\Gamma_n + \omega\varepsilon_2/v}{\lambda_n^2 + (\omega/v)^2(1-\beta^2\varepsilon_1)} - \frac{\varepsilon_2\Gamma_n + \omega\varepsilon_1/v}{\lambda_n^2 + (\omega/v^2)(1-\beta^2\varepsilon_2)}\right]. \qquad (1.6)$$

Equations (1.2), (1.5), and (1.6) fully determine the field in the waveguide.

2. From the formulas just obtained it is easy to calculate the flux of the Poynting vector due to the free field. Using the orthogonality of the eigenfunctions and going over to integration over positive frequencies, we obtain

$$S^{\pm} = 4e^2\sum_{n=1}^{\infty}\lambda_n^2|\psi_n(M_0)|^2\operatorname{Re}\int_0^{\infty}\frac{|A_n|^2\Gamma_n}{\varepsilon_{1,2}\omega}\,d\omega, \qquad (2.1)$$

where the symbol ± indicates the direction of wave propagation.

It is known, however, that in the waveguide there can exist, at certain frequencies, damped waves which, generally speaking, can produce in our case a unique "blocked" field at the separation boundary, i.e., a field consisting of standing exponentially-damped waves, the lifetime of which is determined by the time of dissipation of the electromagnetic energy in the substance. To clarify the physical aspect of this problem, let us compare the work performed by the electromagnetic field on the particle with the energy flux of this field through a waveguide section located at infinity (we disregard the Cerenkov radiation of the charge). For this purpose we use the Poynting theorem

$$W_\infty - W_{-\infty} = \int_{-\infty}^{\infty} dt \int \mathbf{E}\mathbf{j}\, dV - \frac{c}{4\pi}\int_{-\infty}^{\infty} dt \int [\mathbf{E}\times\mathbf{H}]_n\, dS, \quad (2.2)$$

where the surface integral extends over the waveguide section at infinity, and the volume integral over the waveguide segment between them. From (1.2) and (1.6) we can calculate the first term in (2.2), which represents the work done by the field on the particle

$$\int_{-\infty}^{\infty} dt \int \mathbf{E}\mathbf{j}\, dV = S^+ + S^- - \frac{4e^2}{v}\sum_{n=1}^{\infty} \lambda_n^2 |\psi_n(M_0)|^2$$

$$\times \int_0^\infty \frac{(\varepsilon_1-\varepsilon_2)\,[\lambda_n^2 + (\omega^2/v^2)(1-\beta^2(\varepsilon_1+\varepsilon_2))]\, d\omega}{\varepsilon_1\varepsilon_2\,[\lambda_n^2 + (\omega^2/v^2)(1-\beta^2\varepsilon_1)]^2\,[\lambda_n^2 + (\omega^2/v^2)(1-\beta^2\varepsilon_2)]^2}. \quad (2.3)$$

It can be readily shown that the sum in (2.3) represents the energy flux due to the field connected with the particle. Furthermore, (2.2) contains terms for the energy flux due to interference between the field of the particle proper and the radiation field. These terms have the form

$$\int_0^\infty \Phi_1(\omega)\exp\{i(\omega/v + \Gamma_n)z\}\, d\omega,$$

$$\int_0^\infty \Phi_2(\omega)\exp\{i(\omega/v - \gamma_n)z\}\, d\omega. \quad (2.4)$$

An estimate of these integrals at large z, for example by the well-known stationary-phase method, shows that the first of these terms diminishes exponentially with increasing $|z|$, while the second diminishes as $z^{-1/2}$. This result is physically understandable, since in the former case the particle field has practically all passed into the second medium, while in the latter case the radiation and the particle move in one direction and interfere intensely over a certain distance, until the field overtakes the particle. This distance is the characteristic zone of formation of transition radiation. More detailed estimates will be given below.

The foregoing leads to the conclusion that all the particle losses (we do not consider polarization losses) go into production of traveling waves. There is no localized field in the direct vicinity of the separation boundary. It must be recalled, however, that in transition radiation in a slab, this field, generally speaking, exists and must be taken into account.

3. Let us consider radiation for ultrarelativistic incident particles and for the simplest dependence of ϵ_i on the frequency, of the form $\epsilon_1 = 1$ and $\epsilon_2 = 1 - (\omega_0/\omega)^2$, where $\omega_0 = \sqrt{4\pi e^2 N/m}$ is the plasma frequency of the medium.

We estimate first the dimensions of the zone of formation of the transition radiation. The particle field will interact for the longest time with that group of frequencies, the wave packet of which moves at a group velocity v. The center of this packet is at the frequency

$$\omega_n = c\sqrt{(\lambda_n^2 + \omega_0^2/c^2)/(1-\beta^2)}.$$

The interaction time $\Delta\tau_n$ is determined by the time of dissolution of this packet by dispersion in the waveguide; its order of magnitude is

$$\Delta\tau_n = 1/c\sqrt{(1-\beta^2)(\lambda_n^2 + \omega_0^2/c^2)},$$

and therefore the dimensions of the effective zone of formation of transition radiation are given by

$$l_n \approx 1\Big/\sqrt{(1-\beta^2)(\lambda_n^2 + \omega_0^2/c^2)}. \quad (3.1)$$

The same result is obtained when the method of stationary phase is applied to (2.4).

Equation (3.1) determines the dimensions of the zone in which the motion of the particle cannot be distrubed. For example, we can draw from this the qualitative conclusion that to obtain a maximum effect in a slab (i.e., in order for the effects on the front and rear sides to add up) it is necessary to have a slab of thickness $\delta \gg l_1$.

In this case we can also calculate the total radiation energy. Integrating with respect to ω in (2.1) we obtain

$$S^- \approx 0,$$

$$S^+ = \frac{e^2\pi}{\sqrt{1-\beta^2}}\sum_{n=1}^{\infty}|\psi_n(M_0)|^2$$

$$\times\left[\frac{1}{\lambda_n} + \frac{\lambda_n^2}{(\lambda_n^2+k_0^2)^{3/2}} - \frac{4\lambda_n^2}{k_0^2}\left(\frac{1}{\lambda_n} - \frac{1}{\sqrt{\lambda_n^2+k_0^2}}\right)\right], \quad (3.2)$$

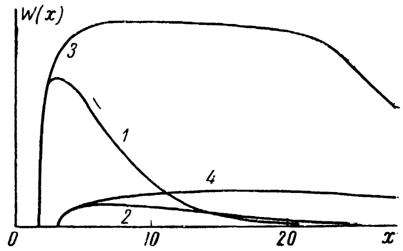
where $k_0 = \omega_0/c$.

It is seen from (3.2) that the principal radiation is along the motion of the particle and is proportional to its energy. This circumstance was pointed out by Garibyan.[3]

Going to the limit in (3.2), by increasing the transverse dimensions of the waveguide without bounds, the sum becomes an integral, evaluation of which leads to

$$S^{+} = e^2\omega_0 / 3c\sqrt{1-\beta^2}, \quad (3.3)$$

which coincides with Garibyan's formula.[3]



Radiation energy density as a function of a dimensionless parameter. 1 – E_{01} mode and 2 – E_{02} mode at 5 Mev particle energy, 3 – E_{01} and 4 – E_{02} mode at 50 Mev particle energy.

The diagram shows, on an arbitrary scale, curves for the radiation energy density W_n vs. the dimensionless parameter $x = \omega/\omega_0$ for a round waveguide, at $n = 1$ and 2 (E_{01} and E_{02} modes) and $\omega_0 r_0/c = 1.85$ (r_0 is the waveguide radius), for incident particles with energies on the order of 5 and 50 Mev. A comparison of the curves shows that the E_{02} mode is vanishingly small compared with the E_{01} mode (the scale of E_{02} is magnified ten times in the figure).

The radiation is essentially concentrated in the region of frequencies of order

$$c\sqrt{(\lambda_n^2 + k_0^2)/(1-\beta^2)};$$

while the magnitude of the radiated energy at a fixed frequency depends relatively little on the particle energy.

The result obtained shows that transition radiation can apparently be used to measure the energy of ultrafast particles. Bearing in mind the possibility of the use of transition radiation to generate millimeter waves, it becomes necessary, if noticeable radiation power is to be obtained, to use particle bunches whose dimensions are much less than the radiated wavelength (see reference 4 and the literature therein).

Assume for example, $\nu = 6 \times 10^9$ particles in the bunch, a bunch repetition rate 10^7/sec ($J = 10$ ma), $\omega \simeq 7.8 \times 10^{11}$ sec^{-1}, and $\Delta\omega/\omega \approx 0.1$. Then at particle energies on the order of 5 Mev the power radiated in the E_{01} mode is on the order of 15 watts. We also note that to increase the radiation efficiency it is advisable to use a series of alternating dielectric plates.

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