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SIMPLE METHOD FOR COMPUTING THE MEAN RANGE OF RADIATION IN IONIZED GASES AT HIGH TEMPERATURES

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The optical absorption coefficients for multiply-ionized high-temperature gases are considered. A simple method is given by which it is possible to approximate rapidly the range of radiation averaged over the optical spectrum. This range determines the radiative thermal conductivity and emissivity of a gas at different temperatures and densities.

1. In the analysis of high-temperature phenomena it is usually necessary to consider the roles of radiative heat exchange and general radiation loss in determining the energy balance of a heated body. To determine these quantities, the mean range of the radiation must be known.

When the density of the gas is not too high the optical photons are generally absorbed in bound-free transitions of electrons from ground and excited states of ions (and atoms) and in free-free transitions. Bound-free transitions are not important energetically because of their small line widths.

At temperatures of the order of tens of thousands of degrees and higher, when the atoms are multiply ionized, the absorption is due to ions of different kinds. Hence, in computing the range for a given temperature and density it is necessary to determine the concentrations of various ions; this procedure involves the solution of a system of algebraic equations for ionization equilibrium, a task which is rather laborious. The labor involved is clearly not justified when one considers the fact that the "hydrogenic" approximation is generally used for establishing the effective absorption cross sections in complicated ions (and atoms); this approximation gives results which are probably never better than to within an order of magnitude.

Below we present a simple, rapid approximation method which can be used to estimate the mean range of radiation in any gas.

2. The total absorption factor for optical radiation of frequency ν by atoms which are ionized m times (m-ions) (bound-free) and by the field of (m+1)-ions (free-free) is[1]

$$\varkappa_{\nu m} = aN_m (m+1)^2 T^{-2} e^{-x_{1m}} F_m(x), \tag{1}$$

where

$$a = (16\pi^2/3\sqrt{3})\, e^6/chk = 0.96\cdot 10^{-7}\ \text{cm}^2\text{-deg}^2, \tag{2}$$

$$x = h\nu/kT, \qquad x_{1m} = I_{m+1}/kT, \tag{3}$$

I_{m+1} is the ionization potential of the m-ion and N_m is the number of m-ions in one cubic centimeter. The frequency dependence is given by the factor

$$F_m(x) = x^{-3}\Big[2x_{1m}\sum_n n^{-3}\exp(x_{1m}/n^2) + 1\Big]. \tag{4}$$

The summation is taken over all levels with principle quantum number n from which a photon $h\nu$ can eject an electron, i.e., $x_{1m}/n^2 < x$.

As has been indicated by Unsöld,[1] because the field in non-hydrogenic atoms and ions is not a true Coulomb field, each level characterized by the number n is split into $2n^2$ levels. For this reason we can replace the summation over n by integration from the lower limit $x = x_{1m}/n^2$:

$$F_m(x) \approx x^{-3}e^x, \qquad x < x_{1m}. \tag{5}$$

Equation (5) does not hold when $x > x_{1m}$, because all levels contribute to the absorption and the summation in (4) yields a constant. The dominant role in this case is played by the ground level n = 1; hence we can write

$$F_m(x) \approx 2x_{1m}x^{-3}e^{x_{1m}}, \qquad x > x_{1m}. \tag{6}$$

The total absorption factor κ_ν is obtained if we sum $\kappa_{\nu m}$ over all ions, i.e., over m (for simplicity we assume that the gas consists of atoms of one element only).

The radiative thermal conductivity is determined from the Rosseland mean-free path defined by

$$l = \int_0^\infty \frac{G(x)\,dx}{\varkappa_\nu(1-e^{-x})}, \qquad G(x) = \frac{15}{4\pi^4}\frac{x^4e^{-x}}{(1-e^{-x})^2}. \tag{7}$$

The integrated emissivity of the gas is determined from the mean absorption coefficient

$$\varkappa_1 = 1/l_1 = \int_0^\infty \varkappa_\nu(1-e^{-x})\,G_1(x)\,dx,$$

$$G_1(x) = 15\pi^{-4}x^3/(e^x-1) \tag{8}$$

3. Substituting κ_ν from (5) and (6) in (8) and computing the integral, we have

$$\varkappa_1 = 1/l_1 = (45a/\pi^4 T^2)\sum_m N_m (m+1)^2 x_{1m} e^{-x_{1m}}. \quad (9)$$

In the expression for the Rosseland mean free path in (7) we have

$$l = \frac{T^2}{a}\int_0^\infty \frac{G(x)(1-e^{-x})^{-1}\,dx}{\sum_m N_m (m+1)^2 e^{-x_{1m}} F_m(x)}. \quad (10)$$

We cannot avoid integration over the spectrum, as in the case of κ_1, since here we do not average the absorption coefficient itself, but its reciprocal.

According to (5), all ions have the same absorption in their transmission bands, [for $x < x_{1m}$ ($h\nu < I_{m+1}$)]. The upper limit on the integral in (10) is the lowest transmission limit associated with ions which are present in sufficient number to make an important contribution in the summation over m.

At a given temperature and density the gas contains appreciable numbers of ions of two or three kinds. If the densities are not too high these ions have ionization potentials which are much higher than kT, so that their transmission limits x_{1m} are at the limits of the region of the spectrum which gives the main contribution in the integral in (10) [the maximum of the weighting function G(x) occurs at $x \approx 4$].

For this reason, as an approximation we can neglect the dependence of $F_m(x)$ on m, the charge of the ion, and take this function out from under the summation in (10); also, we can extend the expression in (5) to the region $x > x_{1m}$ (the integral still converges rapidly). The integral which results has been computed by Unsöld[1] and its value is 0.87. We have

$$l = (0.87T^2/a)\left\{\sum_m N_m (m+1)^2 e^{-x_{1m}}\right\}^{-1}. \quad (11)$$

4. The summations over m in (9) and (11) are carried out by a method which has been used earlier.[2]

We regard the ion concentration N_m and the ionization potential I_m as continuous functions of the number m and replace the summation in (9) and (11) by integration.*

The ion distribution function N(m) has a narrow peak whose skirts fall off more rapidly than the quantity $\exp\{-x_1(m)\}$ increases with decreasing m (cf. Appendix). Hence we can take out the mean value of the factors as averaged over the distribution function N(m). Then

$$l_1 = (\pi^4/45a)(T^2/N)\exp(\bar{x}_1)/(\bar{m}+1)^2\bar{x}_1,$$
$$l = (0.87/a)(T^2/N)\exp(\bar{x}_1)/(\bar{m}+1)^2, \quad (12)$$

where $N = \sum_m N_m$ is the number of original atoms per cubic centimeter. The mean charge $\bar{m}$ (obviously equal to the number of free electrons in the atom) and the potential $\bar{x}_1$ are determined in the same approximation from the Saha equations for N_m. As has been shown in reference 2, $\bar{m}$ can be found from the elementary transcendental equation

$$\bar{x}_1 = I(\bar{m}+1/2)/kT = \ln(AT^{3/2}/N\bar{m}),$$
$$A = 2(2\pi m_e k/h^2)^{3/2} = 4.8\cdot 10^{15}\ \mathrm{cm}^{-3}\,\mathrm{deg}^{-3/2}. \quad (13)$$

Substituting (13) in (12) and introducing the numerical values of the constants, we obtain the final formulas for the mean ranges:

$$l_1 = 1.1\cdot 10^{23} T^{7/2}/N^2\bar{m}(\bar{m}+1)^2 x_1\ \mathrm{cm},$$
$$l = 4.4\cdot 10^{22} T^{7/2}/N^2\bar{m}(\bar{m}+1)^2\ \mathrm{cm}. \quad (14)$$

5. An idea of the numerical values of the ranges and the mean values of the ion charges and potentials, can be obtained from the Table, which shows the results of a calculation for air.* In the region of T and N being considered the dependence of range on these quantities is given roughly by the relations

$$l \sim T^{1.35} N^{-1.80}, \qquad l_1 \sim T^{1.35} N^{-1.68}.$$

T, deg		N/N_{norm}		
		1	10^{-1}	10^{-2}
50 000	$\bar{m}$	1.4	1.85	2.35
	$\bar{x}_1$	6.65	8.6	10.7
	l, cm	0.0053	0.28	17
	l_1, cm	0.002	0.08	3.9
100 000	$\bar{m}$	2.72	3.47	4.1
	$\bar{x}_1$	6.52	8.85	10.7
	l, cm	0.013	0.7	47
	l_1, cm	0.005	0.2	11

As an example of an emissivity calculation we find the radiative cooling rate of transparent (in the sense that $R \ll l_1$) air at T = 50,000° and density $N/N_{\mathrm{norm}} = 10^{-2}$. We have $j = 4\sigma T^4/l_1 = 3.6 \times 10^{14}$ erg/cm³·sec. The internal energy under these conditions is $\epsilon = 83$ ev/atom,[2] so that the initial cooling rate is $\epsilon/j = 1.9 \times 10^{-7}$ sec.

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*The function I(m) is plotted graphically by drawing a continuous curve through the discrete points (I_m) taken from appropriate tables.[3] The curves meet at the point I(0) = 0 [the ionization potential of the neutral atom is denoted by $I_1 = I(1)$].

*The ionization potentials for oxygen and nitrogen are taken as averages corresponding to the percentage content of these components.

The error introduced by the approximation used in calculating the summations in (9) and (11), as can be seen by comparison with the exact values computed from the data of Selivanov and Shlyapintokh[4] on ionization equilibrium in air, is found to be very small. In any case this error is much smaller than the possible errors which arise because of the "hydrogenic" approximation.

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APPENDIX

ION CHARGE DISTRIBUTION IN MULTIPLE IONIZATION

Assuming for simplicity that the ratio of the statistical weights of the electronic states of the ions in the Saha formulas is unity we have

$$N_{m+1}N_e/N_m = 2(2\pi m_e kT/h^2)^{3/2}\exp\{-I_{m+1}/kT\}.$$

Combining the equations for different values of m = 0, 1, 2, ... and using the definition of the "average potential" (13), we have ($\overline{m} = N_e/N$)

$$N_{m+n}/N_m = \exp\Big[-\sum_{l=1}^{n}(I_{m+l}-\bar{I})/kT\Big],$$

$$N_{m-n}/N_m = \exp\Big[-\sum_{l=0}^{n}(\bar{I}-I_{m-l})/kT\Big], \qquad (15)$$

where n = 1, 2, 3, ... We take the value of m for which N_m is a maximum. $\bar{I}$ corresponds approximately to the potential of the same atoms so that all the terms in the summations are positive and the ion concentration falls off on both sides of the maximum.

In Eq. (15) we convert to the continuous functions N(m) and I(m) and write as an approximation:

$$I(m) \approx \bar{I} + (dI/dm)_{\overline{m}+1/2}(m-\overline{m}),$$

obtaining the Gaussian distribution

$$N(m) = N_{max}\exp\Big[-\Big(\frac{m-\overline{m}}{\Delta}\Big)^2\Big],$$

$$\Delta = [2kT/(dI/dm)_{\overline{m}+1/2}]^{1/2} < (2kT\overline{m}/\bar{I})^{1/2} = (2\overline{m}/\bar{x}_1)^{1/2}, \qquad (16)$$

if it is assumed that on the average dI/dm > I/m.

Taking the quantities $\overline{m}$ and $\bar{x}_1$ from the table, we see that the half width of the peak $\Delta \lesssim 1$, i.e., the peak is actually quite narrow.

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[1] A. Unsöld, Physics of the Stellar Atmospheres, IIL (1949) (Russ. Transl.).

[2] Yu. P. Raĭzer, JETP **36**, 1583 (1959); Soviet Phys. JETP **9**, 1124 (1959).

[3] Kaye and Laby, Tables of Physical and Chemical Constants, Longmans, N.Y. 1948, (Russ. Transl. IIL 1949).

[4] V. V. Selivanov, and I. Ya. Shlyapintokh, J. Chem. Phys. (U.S.S.R.) **32**, 670 (1958).

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