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SOME PROBLEMS OF THE DYNAMICS AND HEATING OF A CONDUCTING MEDIUM IN A MAGNETIC FIELD

G. S. GOLITSYN

Institute of Atmospheric Physics, Academy of Sciences, U.S.S.R.

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One-dimensional motion of a perfectly conducting medium under the action of a magnetic field prescribed at the boundary is considered. Confinement of the plasma by a high frequency magnetic field is investigated. Some aspects of the problem of heating the plasma by magnetoacoustic waves are discussed.

IT is well known that a magnetic field acts on a perfectly conducting medium as a sort of piston. The peculiar feature of a magnetic piston compared to pistons encountered in ordinary gas dynamics is that at the boundary of the medium the pressure $H^2/8\pi$ is prescribed, while in gas-dynamic problems the speed of the piston is given.

If the magnetic field is specified at the initial moment inside the medium as well, then this extends the conditions of applicability of the theory under discussion to an actual hot plasma (for details see reference 1), in which in this case magneto-acoustic waves may occur,[2] while acoustic waves are not possible inside a plasma of finite dimensions without a field because of the large mean free path of the ions.

We shall assume that the field within the medium is perpendicular to the direction of motion. In such a case the hydrodynamic equations will hold also for a low-density plasma.[3] Moreover, this case is relatively simple analytically compared to the case of an arbitrary direction of the field. From a gas dynamic point of view the theory of magnetoacoustic waves of finite amplitude is completely analogous to the theory of Riemann waves in ordinary gas dynamics, and differs from it only by the greater complexity of the analytic expressions.[4]

1. MOTION UNDER THE ACTION OF A FIELD PRESCRIBED OVER THE BOUNDARY

The boundary condition at the surface of separation between the medium and the magnetic field may be easily established by choosing the system of coordinates in which this boundary is at rest. We assume that the medium is perfectly conducting, so that there is no flux of the field into the medium nor of the medium into the field. From the condition of the conservation of the normal component of the momentum flux we obtain at the boundary the equality of the total pressures

$$H_e^2/8\pi = p + H_i^2/8\pi. \tag{1.1}$$

Here H_e is the intensity of the external field and H_i is the field inside the medium.

We first investigate isentropic motion, and for simplicity set $H_i = 0$. Then in the case of one-dimensional motion the pressure is determined by the following equation[5]

$$p = p_0[1 \pm (\gamma - 1)u/2c_0]^{2\gamma/(\gamma-1)}, \tag{1.2}$$

where u is the velocity of the medium, and c is the velocity of sound. The subscript 0 denotes quantities defined at points where the gas is at rest. From (1.1) and (1.2) we obtain the following boundary condition:

$$H_e^2/8\pi = p_0[1 \pm (\gamma - 1)U/2c_0]^{2\gamma/(\gamma-1)}. \tag{1.3}$$

Here $U = dx/dt$ is the velocity of the boundary corresponding to given values of the external field and of the pressure of the medium. If the pressure is larger than $H_e^2/8\pi$, then the medium will expand, and a rarefaction wave will travel through it, while when $H_e^2/8\pi$ is greater than the pressure the gas will be compressed. From (1.3) we obtain the differential equation for the motion of the boundary of separation (the magnetic piston):

$$\frac{dx}{dt} = \frac{2c_0}{\gamma - 1}(\eta_e^{(\gamma-1)/2\gamma} - 1), \tag{1.4}$$

where $\eta_e = H_e^2/8\pi p_0$ is a given function of t and, perhaps, also of x. The system of coordinates is chosen in such a way that at the initial time the medium is situated at $x \geq 0$. When motion occurs in the direction $x > 0$ we have $U > 0$. On integrating (1.4) we obtain the law of motion for the piston (for the separating boundary).

By utilizing results obtained earlier,[4] we obtain in exactly the same manner the corresponding equation for the case $H_i \neq 0$ $(\gamma = 5/3)$, which, unfortunately, is not solved with respect to the derivative

$$\eta_e = y^3\left(y^2 + \frac{5}{6}\eta_i\right),$$

$$y = \frac{c}{c_0} = \frac{1}{\eta_i}\left\{\left[(1+\eta_i)^{3/2} \pm \frac{\eta_i}{2c_0}\frac{dx}{dt}\right]^{2/3} - 1\right\}. \quad (1.4')$$

Here $\eta_i = H_{i0}^2/4\pi\rho_0 c_0^2$ = const; η_e has its former meaning. If we know the law of motion for the piston we can in principle completely describe the isentropic motion of the gas by determining the velocity in the resulting simple wave in parametric form (cf. reference 5, section 94, problem 2). Generalization to the case $H_i \neq 0$ presents no difficulties.

Let us examine several examples. Let η_e = const < 1. Then the gas will expand into a vacuum, where there is a magnetic field. The limiting expansion velocity will be constant and in absolute value will be equal to

$$u_{max} = (1 - \eta_e^{(\gamma-1)/2\gamma})\, 2c_0/(\gamma - 1). \quad (1.5)$$

A wave of rarefaction will travel through the gas, as in the case of motion of a real piston with a speed $U = u_{max}$.

Now let η_e = const > 1. Then a shock wave will travel through the gas, compressing the gas to a pressure $p_2 = H_e^2/8\pi$. The boundary of separation will start moving with the velocity of the stream behind the front of the wave determined by the Huygens condition at the discontinuity.

2. ON THE CONFINEMENT OF PLASMA BY A RAPIDLY VARYING MAGNETIC FIELD AND ON THE HEATING OF PLASMA BY ACOUSTIC WAVES

The problem of the confinement of plasma by high frequency fields has been investigated by a number of authors.[1,6,9] Let us consider the one-dimensional problem of the motion of plasma (which we shall describe by means of the equations of magnetohydrodynamics) under the action of a magnetic field at the boundary which varies according to $H_e = H_0 \sin \omega t$. The field should not penetrate deeply into the medium, and this imposes the first condition on the frequency:

$$\omega \gg c^2/4\pi\sigma L^2, \quad (2.1)$$

where $\sigma \approx 2 \times 10^{13}\, T^{3/2}$ (the temperature is given in ev) is the conductivity of the medium, and L is a characteristic dimension of the system. When L = 10^2 cm, T = 100 ev, condition (2.1) yields $f = \omega/2\pi \gg 0.1$ cps. A much more difficult condition from a technical point of view is the second one: the deviations of the separation boundary from its average position must be small in comparison with L, and this can be written in the form

$$c_0/\omega \ll L, \quad (2.2)$$

where c_0 is the velocity of sound in the medium. When T = 100 ev the velocity of sound in a deuterium plasma is of the order of 5×10^6 cm/sec, which in the case of L = 10^2 cm yields $f \gg 10^4$ cps.

The pressure of the magnetic field must balance the time average of the plasma pressure. The problem consists of determining the amplitude of the balancing field, and of making more precise the condition (2.2).

For the sake of simplicity we consider first the case when there is no field within the medium. In this case the motion of the boundary is described by (1.4). We introduce the dimensionless variables: $\omega t = \tau$; $(\gamma - 1)\,\omega x/2c_0 = \xi$. Then (1.4) takes the form

$$d\xi/d\tau = \eta^{(\gamma-1)/2\gamma} - 1, \quad (A)$$

from which we obtain for the case of a sinusoidal field under the condition $\xi = 0$ at $\tau = 0$

$$\xi(\tau) = \int_0^\tau (H_0^2 \sin^2\tau/8\pi p_0)^{(\gamma-1)/2\gamma} d\tau - \tau. \quad (2.3)$$

We require that the average position of the boundary remain unaltered, and for this it is necessary that the equality $\xi(\pi) = 0$ should hold (since the pressure of the field is proportional to the square of H, the period of pressure oscillation at the boundary is equal to π). Due to the fact that the field appears nonlinearly in (2.3) there is no guarantee that the amplitude H_0 is determined by the seemingly obvious condition

$$\overline{H^2}/8\pi = \overline{H_0^2 \sin^2\tau}/8\pi = H_0^2/16\pi = p_0. \quad (B)$$

We therefore write $H_0 = \alpha_\gamma(8\pi p_0)^{1/2}$ and we determine the factor α_γ from the condition $\xi(\pi) = 0$. From this condition, and from (2.3) we obtain

$$\alpha_\gamma = \left[\frac{1}{\pi}\int_0^\pi (\sin\tau)^{(\gamma-1)/\gamma} d\tau\right]^{-\gamma/(\gamma-1)}$$

$$= \left[\frac{\sqrt{\pi}\,\Gamma(1 + (\gamma-1)/2\gamma)}{\Gamma(1/2 + (\gamma-1)/2\gamma)}\right]^{\gamma/(\gamma-1)}. \quad (2.4)$$

The factor α_γ depends on the properties of the gas, but only to a very small extent. We present a small table which shows this dependence for different values of the adiabatic exponent γ:

$\gamma =$	1	$^7/_5$	$^5/_3$	2	3	∞
$\alpha_\gamma =$	2	1.81	1.76	1.72	1.66	1.57

It turns out that the amplitude of the external confining field must be larger by 20 — 30% than one might have supposed on the basis of elementary considerations without a detailed analysis of the motion of the boundary.

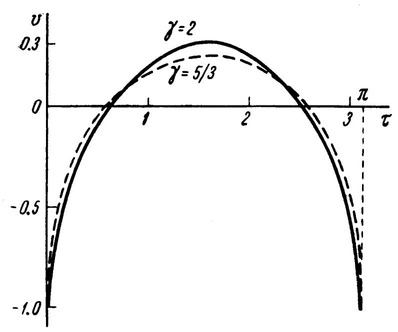

FIG. 1

Figures 1 and 2 give the results of numerical integration leading to the determination of the time dependence of the velocity $v = (\gamma - 1)/2c_0(dx/dt)$, and of the position of the separating boundary for the cases $\gamma = {^5/_3}$ and $\gamma = 2$. The maximum variation in the amplitude of oscillation $\Delta\xi_{max} = \xi_{max} - \xi_{min}$ is in the first case equal to 0.332, and in the second case to 0.396; corresponding to this Δx_{max} is respectively equal to $0.996\,c_0/\omega$ and $0.792\,c_0/\omega$.

Let us now consider the case when a magnetic field parallel to the boundary exists within the medium. In this case one must use Eq. (1.4′), which is quite awkward. However, a simple approximate analysis is possible. In order to utilize it, the actual adiabat of the gas $p = c\rho^\gamma$ must be approximated by the adiabat $p = c_1\rho^2$. This approximation will increase in accuracy as the field within the medium increases, simply because the relative role played by the thermodynamic pressure becomes less important.* In this case the whole of magnetohydrodynamics reduces to ordinary gas dynamics, provided we interpret the pressure to mean the total pressure $p^* = p + H^2/8\pi$, while the velocity of sound is interpreted to mean the velocity of propagation of magnetoacoustic waves $c_m = (c_0^2 + H^2/4\pi\rho)^{1/2}$.

Were a constant magnetic field H_{0i} to exist within the medium, it would apparently be natural to assume the external confining field to have a constant component equal to H_{0i}, and a variable

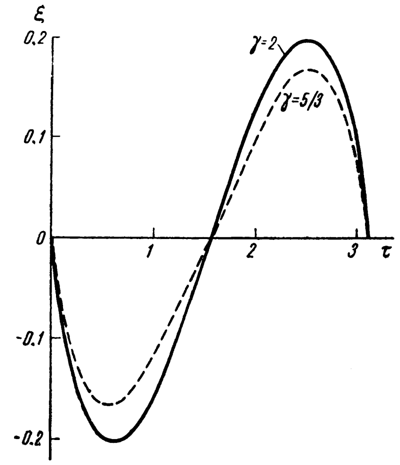

FIG. 2

component which on the average balances the gas pressure of the plasma. Let us investigate the same problem as before. In the case when the field has a constant component the period of pressure oscillation at the boundary will be 2π, and the conditions that the boundary should return to its initial position will be altered: we now must have $\xi(2\pi) = 0$. In the case $\gamma = 2$ this condition as applied to (2.3) may be brought into the form

$$I(a;\ \beta) = 2\pi(1+\beta)^{1/4} = \int_0^{2\pi} (|\,1 + a\sin\tau\,|)^{1/2}\, d\tau,$$

$$\beta = 8\pi p_0 / H_{0i}^2, \qquad a = \alpha\sqrt{\beta}; \tag{2.5}$$

where the factor α is introduced for the same purpose as in the preceding case. In the case of large β (small H_{0i}) we can obtain a correction to the value of α_2 obtained in the preceding case: $\alpha = \alpha_2 + 1.45\,\beta^{-1/2}$. Thus, if the external field has a constant component, the amplitude of the variable field must be larger; this is due to the fact that during the time $\pi < \tau < 2\pi$ the constant and the variable field have opposite signs, and the pressure due to the field during this half period decreases. In the limit of small β (large fields) we obtain by constructing the function $I(a;\beta)$ the limiting value $a = 7.3$, for which we still have

*As is well known,[3] the motion of a low-density plasma across a strong magnetic field is described by hydrodynamic equations with exponent $\gamma = 2$. This result is also obtained in magnetic gas dynamics for adiabatic motion in the case $\eta = H^2/8\pi p \to \infty$ (cf., for example, a number of problems solved in references 4 and 7). However, this does not hold, for example, in the case of shock waves. If the pressure discontinuity across the front is so large that the conditions $p_2/p_1 \gg \eta \gg 1$ hold, then the limiting density discontinuity will be determined by the properties of the gas itself,[4] i.e., by its own value of the exponent γ.

$I(a; 0) = 2\pi$. In this case the external field is given by $H_e = H_0(1 + 7.3 \sin \omega t)$. These calculations show that it is not advantageous to have a constant component in the confining field, but that it is more convenient to have it wholly variable.

Let us discuss the limits of applicability of these results. Rarefaction and compression waves leave alternately the oscillating boundary, with the latter waves eventually turning into shock waves. At the point where the shock wave is formed a discontinuity in entropy also appears, and separates the region of the gas beyond the shock wave from the simple wave directly adjacent to the boundary. Since the discontinuity in the entropy is not displaced with respect to the medium, it will not approach the boundary (except for special values of the piston velocity,[8] where the motion must be accelerated in order that the discontinuity would be formed immediately at the boundary, whereas in our case the motion is periodic). Therefore the basic equation (1.4) will correctly describe the behavior of the boundary until a reflected shock wave arrives at that point. Then the gas compressed by the shock wave will begin to expand, a rarefaction wave will travel through the gas and equalize the pressure, and the oscillations of the boundary will stabilize around some new coordinate. Generally speaking, during the heating process the amplitude of the field must increase, but the ratio between $H_0^2/8\pi$ and p_0 must remain the same as found earlier.

The waves leaving the boundary will carry away a large amount of power, amounting to tens of kilowatts (cf. reference 6). The problem of the heating of plasma by acoustic waves has already been considered previously,[6,9] but without introducing a specific dissipation mechanism. In our case the formation of discontinuities in acoustic (magnetoacoustic) waves will be a more powerful loss mechanism than viscous, thermal, and Joule losses. Since the amplitude of pressure oscillations is great, the discontinuity will be formed already in the first wave. If condition (2.2) holds, we have many waves with shock discontinuities in a length L, and therefore the dissipation is great.

Let us check as to what will be the Larmor frequency of the ions in this case, since magnetoacoustic waves can exist only at frequencies much lower than the ion Larmor frequencies.[2] In the case of deuterium this is equivalent to the inequality

$$f \ll f_L = eH / 2\pi mc = 7.65 \cdot 10^3 H, \tag{2.6}$$

where H is given in oersteds. The inequalities (2.6) and (2.2) are contradictory (in the case $H_{0i} \neq 0$ we should interpret c_0 in (2.2) as c_m), but the technically important range of their simultaneous validity does exist. Apparently the optimum variant from the point of view of satisfying these two inequalities is the case $\eta = 1$.

For the purpose of heating the plasma by a pulsating magnetic field we should investigate the case when the field oscillates in phase at both ends of a plasma region of length 2L. If the amplitudes of the fields are the same, then the picture will be completely symmetric with respect to the central plane, where collision of waves will occur which, in view of the symmetry, can be regarded as reflection from a perfectly rigid wall. The collision of shock waves has also been investigated in magnetohydrodynamics.[4] In this case additional heating and an increase in entropy also take place.

The first shock wave after reflection from the wall and amplification will return, and collide with the shock wave following behind it. In this case the collision will no longer by symmetric, but amplification of both waves will in any case occur, although not to such a degree as in the case of the first collision. In the case of continuous motion of the piston at the boundary a stationary picture for strongly nonlinear waves is apparently impossible. In any case this question requires a separate detailed investigation. Probably one can assert that a standing wave is formed for the fundamental harmonic of the nonlinear waves, while the remaining harmonics will exist in the form of traveling waves. However, in all circumstances it is clear that in such a system the dissipation of energy will be very great (cf., for example, reference 5, section 95, problem 1).

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[1] A. I. Morozov, In the collection of articles, Физика плазмы и проблема управляемых термоядерных реакций (Plasma Physics and the Problem of Controlled Thermonuclear Reactions) **4**, U.S.S.R. Academy of Sciences, 1958, p. 331.

[2] S. I. Braginskiĭ and A. P. Kazantsev, ibid. **4**, p. 24.

[3] Chew, Goldberger, and Low, Proc. Roy. Soc. **A236**, 112 (1956).

[4] G. S. Golitsyn, JETP **35**, 776 (1958), Soviet Phys. JETP **8**, 538 (1959).

[5] L. D. Landau and E. M. Lifshitz, Механика сплошных сред (Mechanics of Continuous Media), GITTL, 1954, Ch. X.

[6] B. A. Trubnikov, loc. cit. in reference 2, p. 309.

[7] G. S. Golitsyn, JETP **34**, 688 (1958), Soviet Phys. JETP **7**, 473 (1958).

[8] K. P. Stanyukovich, Неустановившееся движение сплошной среды. (Unsteady Motion of Continuous Media), GITTL, 1955, Ch. 4.

[9] Gerger, Newcomb, Dawson, Frieman, Kulsrud, and Lenard, Phys. Fluids **1**, 308 (1958).

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