ON THE NUCLEON + CORE MODEL OF THE NUCLEUS WITH A VIBRATIONAL EXCITATION SPECTRUM OF THE CORE

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The nucleon + core model with intermediate coupling between the nucleon and the phonon excitations of the core has been considered in a number of papers[1-5] as a possible way of describing the spectra of odd nuclei in the atomic weight region where even-even nuclei have level spectra similar to that of the vibrational quadrupole excitations of the nuclear surface (phonons). The state of the nucleon + core system is a superposition of nucleon states and core states with various numbers of phonons. The energy levels and the wave functions of the system have been determined approximately by diagonalizing the energy matrix, which is cut off at a certain number of phonons, N. The convergence of the approximations for various cutoff values N is investigated on the simple model of spinless photons, which leads to an energy matrix which retains the main features of the matrix for phonons with spin.

INTRODUCTION

THE level spectrum of even-even nuclei in the regions $60 \leq A \leq 150$ and $190 \leq A \leq 214$ has the characteristic features of the vibrational spectrum of the quadrupole excitations of the nucleus:[1] 1) the ratio of the energies of the second and first excited states, E_2/E_1, is close to the value 2; 2) the reduced probabilities for E2 transitions exhibit a collective character; 3) the intensities of the M1 transitions are below the estimates given by the single-particle model; 4) the reduced probabilities for the "crossing" E2 transitions are smaller than the probabilities for E2 transitions between adjacent levels.

The model of vibrational quadrupole excitations of the nucleus predicts three degenerate states with spins I = 0, 2, and 4 for the second excited level. If the conditions of adiabatic motion of the nucleons of the core and collective motion of the core are violated, this degeneracy is removed. At present three levels are known for Cd_{48}^{110} and Hg_{80}^{198}.

For the case of the adjacent odd nuclei it is natural to try to apply the nucleon + core model for the description of the level spectrum of the nucleus. This model has been considered earlier by a number of authors[1-6] and is now regarded as one of the possible descriptions of the odd nuclei.[7]

In the framework of this model the state of the nucleon + core system is given by a solution of the Schrödinger equation with Hamiltonian[1,3]

$$\hat{H}=\hat{H}_{\text{vibr}}+\hat{H}_{\text{single}}+\hat{H}_{\text{cplg}}. \tag{1}$$

Here

$$\hat{H}_{\text{vibr}}=\tfrac{1}{2}\sum_{\mu} B_2\,(-1)^{\mu}\dot{\alpha}_{2\mu}\dot{\alpha}_{2-\mu}+\tfrac{1}{2}\sum_{\mu} C_2(-1)^{\mu}\alpha_{2\mu}\alpha_{2-\mu}, \tag{2}$$

$$\hat{H}_{\text{cplg}}=-k\sum_{\mu}\alpha_{2\mu}Y_{2\mu}(\theta,\ \varphi),$$

$$\hat{H}_{\text{single}}=-(\hbar^2/2M)\,\Delta+U(r) \tag{3}$$

$\hat{H}_{\text{single}}$ is the Hamiltonian of the nucleon in the field of the nuclear core, U(r). For U(r) one takes either the radially symmetric potential of the Nilsson type[8] or the optical model potential. $\alpha_{2\mu}$ are the deformation parameters of the surface of the nucleus,

$$R(\theta,\ \varphi)=R_0[1+\alpha_0+\sum_{\mu}\alpha_{2\mu}Y_{2\mu}].$$

k is the coupling constant of the interaction of the nucleon with the nuclear surface; it can only be estimated within a particular model of the motion of the nucleon in the deformed nucleus. Thus we have, for a nucleon in a rectangular potential well with depth U_0 and with a deformed surface,[1]

$$\hat{H}_{\text{cplg}}=-U_0R_0\delta(r-R_0)\sum_{\mu}\alpha_{2\mu}Y_{2\mu}(\theta,\ \varphi),$$

after averaging over the state of the nucleon with quantum numbers $nj\mu_j$, we obtain

$$\langle \hat{H}_{\text{cplg}} \rangle = -U_0 R_0^3 \,|\, R_{nj}(R_0) \,|^2 \sum_{\mu} \alpha_{2\mu} Y_{2\mu}(\theta,\ \varphi).$$

Hence

$$k = U_0 R_0^3 \,|\, R_{nj}(R_0) \,|^2 = 40 \text{ to } 50 \text{ Mev}.$$

It follows that the dimensionless coupling parameter $\kappa = k\sqrt{\hbar\omega\ 2C_2}/\hbar\omega$ is not small, and that perturbation theory with respect to κ cannot be applied if $\hbar\omega$ and the stiffness C_2 are small: $C_2 \sim 20 \div 60$ Mev, $\hbar\omega \approx 0.2$ to 0.8 Mev.

One usually seeks the solution of the Schrödinger equation for the system in the form of an expansion in terms of core states $\chi^{\nu}_{\lambda\mu_\lambda}$ (ν is the number of phonons, and λ and μ_λ are the angular momentum and its projection on the axis of quantization):

$$\Psi_{IM} = \sum_{nj\lambda\nu} a^{Inj}_{\nu\lambda} \sum_{\mu_j \mu_\lambda} C^{IM}_{j\mu_j\lambda\mu_\lambda} \psi_{j\mu_j n} \chi^{\nu}_{\lambda\mu_\lambda} \qquad (4)$$

[$\psi_{j\mu_j n}$ is the function of the nucleon in the spherically symmetric field $U(r)$]. The amplitudes $a^{\cdots}_{\cdots}$ are then determined by diagonalizing the energy matrix.

Approximate solutions of the equation $H\psi = E\psi$ were found in references 1, 3, and 5 by cutting off the energy matrix at a certain number of phonons and assuming that the nucleon moment j is a good quantum number. The cut-off was carried out at the values N = 1, 2, and 3. Since the validity of the model depends on the comparison of the experimental data with the results of the model, there naturally arises the question of the accuracy of these approximations and to what extent the obtained results can be used in the analysis of the experimental data.

THE NUCLEON + CORE MODEL WITH SPINLESS PHONONS

The investigation of the convergence with respect to N for matrices with phonons with spin is difficult, because the state ψ_{IM} for a given number of phonons ν has contributions from several states with different λ. The rank of the matrix is, therefore, considerably greater than N. For this reason we consider the process of approximation on the simple model of spinless phonons. The energy matrix in this model, however, retains the basic features of the matrix for phonons with spin. As model Hamiltonian we take

$$\hat{H}_0 = \tfrac{1}{2}(\hat{q}\,\hat{q}^+ + \hat{q}^+\hat{q}) - \varkappa(\hat{q}^+ + \hat{q}), \qquad (5)$$

i.e., we assume that the state of the "nucleon" does not change. This is equivalent to the assumption that n and j are good quantum numbers. Here $\hat{q}^+$ and $\hat{q}$ are the creation and annihilation operators of the "phonons" and κ is a dimensionless parameter which characterizes the coupling of the nucleon degree of freedom with the "phonons". The exact solution for the Hamiltonian (5) is known: we introduce the new phonon operators $\hat{p} = \hat{q} - \kappa$ and $\hat{p}^+ = \hat{q}^+ - \kappa$, and obtain

$$\hat{H}_0 = \tfrac{1}{2}(\hat{p}^+\hat{p} + \hat{p}\hat{p}^+) - \varkappa^2, \qquad E_i = \hbar\omega\varepsilon_i = \hbar\omega[i + \tfrac{1}{2} - \varkappa^2].$$

We are, however, not interested in the exact value of the energy level E_i, but in the process of approximating E_i by cutting off the energy matrix at the number of phonons N.

We therefore expand the Ψ_i function of the system into a series in terms of states of the core:

$$\Psi_i = \sum_{\nu} a^i_{\nu}\chi_{\nu}, \text{ where } \chi_{\nu} = (\hat{q}^+)^{\nu}\chi_0/\sqrt{\nu!}\,,$$

χ_0 is the "vacuum" wave function.

Substituting the series $\Psi = \sum_{\nu} a_{\nu}\chi_{\nu}$ in the Schrödinger equation $H_0\Psi = E\Psi$, we obtain the energy matrix

$$(\nu - \varepsilon)\,a_{\nu} - \varkappa\sqrt{\nu+1}\ a_{\nu+1} - \varkappa\sqrt{\nu}\,a_{\nu-1} = 0; \qquad (6)$$

here the factor $\frac{1}{2}$ is included in ϵ, so that for $\kappa = 0$ (free oscillator), $\epsilon_\nu = \nu$. We regard equation (6) as a recurrence relation for the coefficients a_ν. Choosing a_0 as the independent coefficient, we find all a_ν successively in terms of a_0:

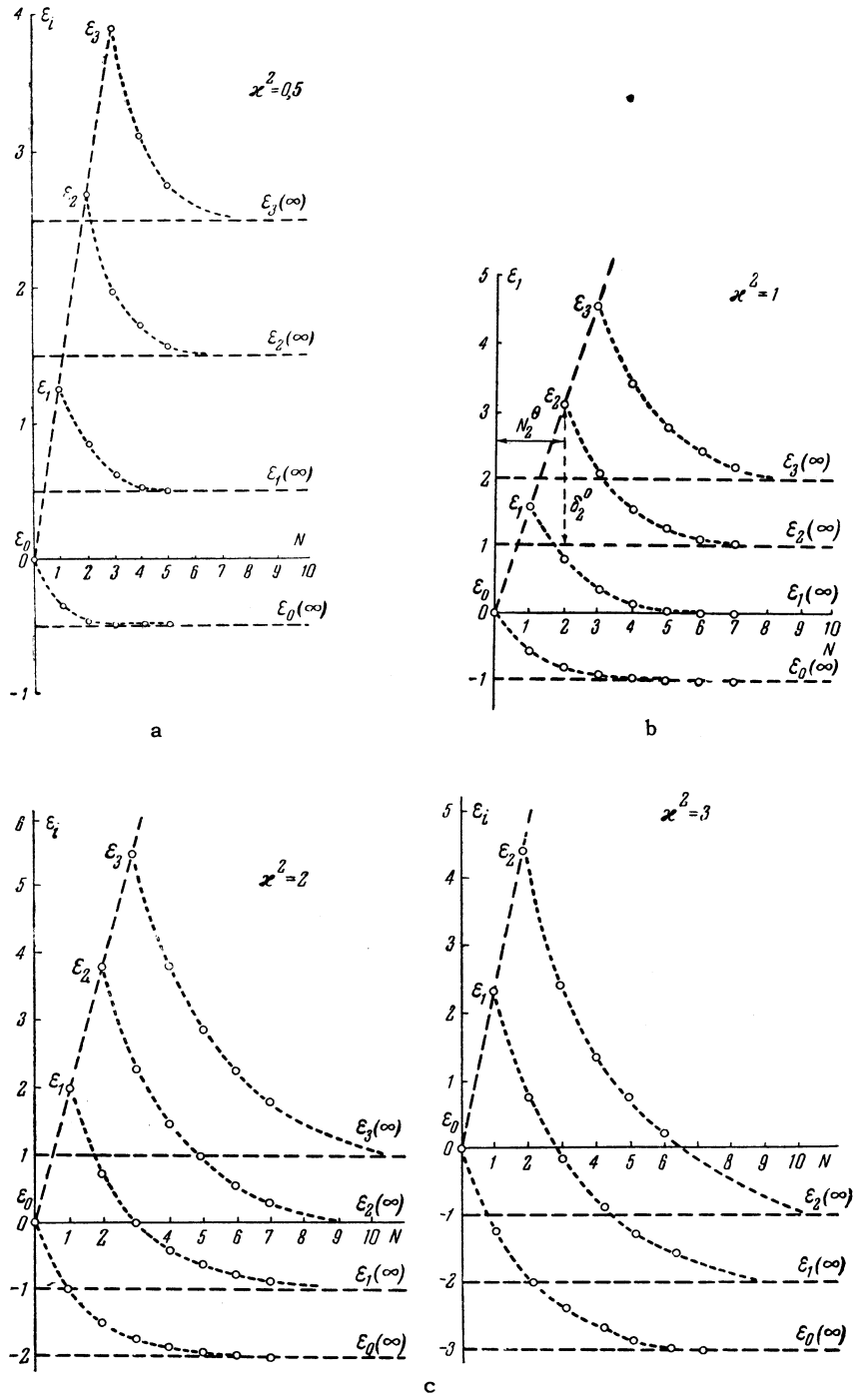
$$a_{\nu} = a_0 f_{\nu}(\varepsilon)/(-\varkappa)^{\nu}\sqrt{\nu!}, \qquad (7)$$

where

$$f_{\nu+1}(\varepsilon) = (\varepsilon - \nu)\,f_{\nu} - \nu\varkappa^2 f_{\nu-1};$$
$$f_1(\varepsilon) = \varepsilon. \qquad (8)$$

Cutting off the matrix (6) at a certain phonon number N corresponds to the equation $a_{N+1} = 0$ or $f_{N+1} = 0$. From this we find N+1 roots, $\epsilon_i(N)$, which are the eigenvalues of the Hamiltonian in the given approximation. For each root $\epsilon_i(N)$ we find the set of amplitudes $a^i_\nu(N)$ which determines the wave function of the i-th state. As is seen from (8), the essential coupling parameter is not κ, but κ^2; we therefore carried out calculations for $\kappa^2 = 0.5$, 1, 2, and 3. The roots $\epsilon_i(N)$ for each κ^2 were found graphically by constructing the corresponding polynomial $f_{N+1}(\epsilon)$. The results of the calculation of the energy levels for the ground and first three excited levels are given in the figure.

To illustrate the convergence process with respect to N, we list in Tables I to IV the amplitudes of the wave functions of the system in the first excited and ground states for the parameter values $\kappa^2 = 1$ and $\kappa^2 = 3$. The calculations were



Dependence of the energy levels ε_i on the number of phonons N for various values of κ^2.

carried out with slide rule accuracy. As is seen from the tables, the coefficients of the wave functions have appreciable magnitudes in the interval N = 1 to 5. The $a_\nu^i(N)$ become smaller by an order of magnitude only for $\nu \approx 5$. The solution given by the cut-off matrix can therefore only be close to the actual solution for $N \approx 5$ or 6. The errors in the calculation of the matrix elements of operators with functions obtained from an energy matrix cut off at $N \leq 3$ may be large, especially in the case of operators which are proportional to $\hat{q}\hat{q}$, $\hat{q}^+\hat{q}^+$, $\hat{q}\hat{q}^+$ and higher powers of the operators $\hat{q}$ and $\hat{q}^+$.

The approximation $\epsilon_i(N)$ to the exact value $\epsilon_i(\infty)$ can, at least in the region $\Delta N \leq 5$, be expressed by the exponential function

$$\varepsilon_i(N) - \varepsilon_i(\infty) = \delta_i(N) = \delta_i^0 \exp\{-\alpha_i(N - N_i^0)\}; \quad (9)$$

N_i^0 and δ_i^0 are indicated in the figure for $\kappa^2 = 1$. For δ_i^0 we have the relation

TABLE I. Amplitudes $a_\nu^0(N)$ for ϵ_0 with $\kappa^2 = 1$

N	ε_0	a_0	a_1	a_2	a_3	a_4	a_5
1	—0.62	0.85	0.53	—	—	—	—
2	—0.86	0.72	0.62	0.31	—	—	—
3	—0.96	0.65	0.62	0.40	0.17	—	—
4	—0.99	0.63	0.62	0.42	0.22	0.067	—
5	—1.0	0.60	0.60	0.43	0.25	0.12	0.054

TABLE II. Amplitudes $a_\nu^0(N)$ for ϵ_1 with $\kappa^2 = 1$

N	ε_1	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	1.62	0.53	—0.85	—	—	—	—	—
2	0.77	0.66	—0.51	—0.56	—	—	—	—
3	0.32	0.68	—0.22	—0.58	—0.39	—	—	—
4	0.14	0.66	—0.09	—0.52	—0.49	—0.25	—	—
5	0.05	0.63	—0.03	—0.47	—0.50	—0.33	—0.13	—
6	0.013	0.61	—0.008	—0.47	—0.50	—0.37	—0.22	—0.095

TABLE III. Amplitudes $a_\nu^0(N)$ for ϵ_0 with $\kappa^2 = 3$

N	ε_0	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	—1.3	0.91	0.40	—	—	—	—	—
2	—2.0	0.81	0.54	0.19	—	—	—	—
3	—2.44	0.73	0.59	0.31	0.13	—	—	—
4	—2.70	0.68	0.61	0.37	0.17	0.054	—	—
5	—2.86	0.62	0.60	0.39	0.20	0.084	0.025	—
6	—2.92	0.62	0.60	0.42	0.23	0.10	0.04	0.012

TABLE IV. Amplitudes $a_\nu^1(N)$ for ϵ_1 with $\kappa^2 = 3$

N	ε_1	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	2.3	0.79	—0.61	—	—	—	—	—
2	0.70	0.94	—0.22	—0.24	—	—	—	—
3	—0.20	0.96	0.065	—0.21	—0.12	—	—	—
4	—0.86	0.95	0.27	—0.10	—0.13	—0.06	—	—
5	—1.30	0.91	0.39	0.00	—0.11	—0.08	—0.03	—
6	—1.54	0.87	0.44	0.061	—0.16	—0.14	—0.07	—0.02

TABLE V

κ^2	α_0	α_1	α_2	α_3	$\bar{\alpha}$	tg φ
1.0	0.80	0.75	0.70	0.60	~0.70	1.50
1.5	0.75	0.65	0.55	0.50	~0.60	1.75
2.0	0.70	0.55	0.45	0.40	~0.50	1.90
3.0	0.55	0.40	0.40	—	~0.45	2.20

$$\delta_i^0 = \varkappa^2 + N_i^0 \tan \varphi(\varkappa^2). \qquad (10)$$

The values of $\alpha_i(\kappa^2)$ and $\tan \varphi(\kappa^2)$ are listed in Table V ($\bar{\alpha}$ is the average value of the parameter α_i over the four energy levels). In view of the crudeness of the calculation of the energy levels, the values of α_i and $\tan \varphi$ reflect only qualitatively the behavior of the energy matrix.

It follows from Table V that the convergence of the energy matrix with respect to the number of included phonons, N, becomes worse as κ^2 and the index of the level, i, increase, since the exponent $\alpha_i(\kappa^2)$ decreases while δ_i^0 becomes larger (the error in the term E_0 of the energy reaches the order $\hbar\omega/2$ for cutoff at $N = 3$ and $\kappa^2 = 3$).

This simple model of phonons without spin permits us to make some qualitative remarks about the more realistic model with phonons with spin

as well, since the energy matrix retains its oscillatory character even in this case.

For simplicity we restrict ourselves to a model in which the state of the nucleon does not change, i.e., the principal quantum number n and the total angular momentum j of the nucleon are good quantum numbers. The state of the system is described by the function

$$\Psi_{IM} = \sum_{\nu\lambda} a^{Ijn}_{\nu\lambda} \sum_{\mu_j \mu_\lambda} C^{IM}_{j\mu_j \lambda\mu_\lambda} \psi_{jn\mu_j} \chi^{\nu}_{\lambda\mu_\lambda}. \quad (11)$$

For the matrix elements of the operator $\hat{H}_{\text{cplg}}$ of (3) (see, for example, reference 3) we have, after some transformations,

$$\langle j,\ \nu\lambda,\ IM | \hat{H}_{\text{cplg}} | j, \nu'\lambda',\ IM\rangle$$

$$= -k \sqrt{\frac{\hbar\omega}{2C_2}\frac{5}{4\pi}} [C^{l0}_{20l0} u\,(2lj^1/_2;\ lj)]$$

$$\times \{\langle \nu\lambda \| \beta \| \nu'\lambda'\rangle u\,(Ij\lambda'2;\ \lambda j)\} \sqrt{\nu+1}\,\delta_{\nu+1,\ \nu'}. \quad (12)$$

Here $u(\alpha\beta\gamma\delta;\ \epsilon\varphi)$ is the Racah function (see, for example, reference 9). The reduced matrix element $\langle \nu\lambda \| \beta \| \nu'\lambda' \rangle$ has the order of magnitude $\sim \frac{1}{2}$ to 1.

The effective coupling constant between the phonons and the nucleon is given by

$$\varkappa_{\text{eff}} = \varkappa \{C^{l0}_{20l0} u\,(2lj^1/_2;\ lj)\}, \quad (13)$$

where

$$\varkappa = k \sqrt{(5/4\pi)(\hbar\omega/2C_2)}/\hbar\omega,$$

$$\{C^{l0}_{20l0} u\,(2lj^1/_2;\ lj)\}^2$$

$$= \begin{cases} l(l+2)(2l-1)/(2l+3)(2l+1)^2 & \text{for } j = l + 1/2, \\ (l^2-1)/(2l+1)^2 & \text{for } j = l - 1/2. \end{cases}$$

Introducing the matrices

$$\{\hat{\gamma}_{\nu,\nu+1}\}_{\lambda\lambda'} = \langle \nu\lambda \| \beta \| \nu+1\lambda'\rangle u\,(Ij\lambda'2;\ j\lambda),$$

$$\{\hat{\gamma}_{\nu\nu-1}\}_{\lambda\lambda''} = \langle \nu-1\lambda'' \| \beta \| \nu\lambda\rangle\ u\,(Ij\lambda2;\ j\lambda'')$$

and the state vector a_ν with components $a_{\nu\lambda}$, we write the energy matrix in a form similar to equation (6):

$$(\nu-\varepsilon)\,a_\nu - \varkappa_{\text{eff}} \sqrt{\nu+1}\,\hat{\gamma}_{\nu,\nu+1} a_{\nu+1}$$

$$- \varkappa_3 \sqrt{\nu}\,\hat{\gamma}_{\nu,\nu-1} a_{\nu-1} = 0. \quad (14)$$

The analysis of this matrix is considerably more difficult than in the case (6). However, utilizing the similarity in the forms of the matrices (14) and (6), we can apply the results obtained earlier for qualitative estimates in this case.

Let us consider the particular nucleus Cd^{111}_{48} (reference 6). For the even-even nucleus Cd^{110}_{48} we have $\hbar\omega = 0.656$ Mev and $C_2 = 58$ Mev; the odd nucleon can be in the states $3s_{1/2}$, $2d_{3/2}$, $2d_{5/2}$, $1g_{7/2}$, and $1h_{11/2}$. The corresponding constants κ^2_{eff} are equal to:

States:	$3s^1/_2$	$2d^3/_2$	$2d^5/_2$	$1g^7/_2$	$1h^{11}/_2$
κ^2_{eff}	0	1.60	1.80	2.5	2.7

The number of phonons, N, to be included in the energy matrix is determined by the accuracy required for the position of the energy terms.

If the admissible error in the position of the term is equal to Δ, the required number of phonons can be estimated from the relation

$$\Delta/\hbar\omega \geqslant \kappa^2_{\text{eff}} \exp(-\bar{\alpha}N). \quad (15)$$

For κ^2_{eff} we must take the largest among all possible values.

If j is not a good quantum number, all nucleon states with the same parity are combined into a single energy matrix. The convergence of this matrix with respect to the included number of phonons is determined by the largest value of the coupling constant for nucleon states with the same parity. In the case of Cd^{111}_{48} the largest coupling constant for even states is $\kappa^2_{\text{eff}} = 2.5$. This constant corresponds to the exponent $\bar{\alpha} \approx 0.5$. If the spectrum is to have an accuracy of the distance to the first excited level, $E_1 = 0.247$ Mev, we must include at least four phonons.

If the accuracy is increased ($\Delta = 80$ kev, $N \geq 6$; and for $\Delta = 20$ kev, $N \geq 9$), the number of phonons is higher.

In the case of $Cd^{110,111}$ we used the values for $\hbar\omega$ and C_2 quoted in the well known review article of Bohr, Mottelson, et al.[6] These values were obtained by fitting the experimental data on the Coulomb excitation of the nuclei in the framework of the hydrodynamic theory of vibrational excitations of the nucleus. On the whole, the hydrodynamic theory contradicts the experimental results, so that the question of the structure of the vibrational excitations of the nuclei remains open. The abovementioned example of $Cd^{110,111}$ must, therefore, be regarded only as an illustration of the convergence process for the energy matrix with respect to the number of phonons.

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