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ON THE QUESTION OF CRITICAL VELOCITIES FOR FLOW OF He II *IN CAPILLARIES*

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As is known, Onsager's[1] and Feynman's[2] ideas about vortex lines yield the right order of magnitude of critical velocities for superfluid helium rotating in a cylinder and for flow from a narrow capillary into a large beaker. In the former case the vortex lines are straight and parallel to the cylinder axis, while in the latter they are rings formed in the beaker near to the junction with the capillary.[2]

It will be shown below that similar values of the critical velocities can be calculated for the flow of helium through a long capillary. It is natural to suppose that the vortex lines will, in this case, be closed curves lying in planes perpendicular to the capillary axis. The shape of the lines will be determined by the capillary cross section, i.e., for a circular cross section the lines will be circular and for a rectangular section the lines will form closed curves nearly rectangular in shape. The angular momentum associated with such lines is evidently zero, while the linear momentum is non-zero and is directed parallel to the capillary axis, i.e., parallel to the flow velocity v. According to Landau[3] the change in energy, ΔE, of flowing helium (in a coordinate system fixed with respect to the capillary walls), associated with the formation of a vortex line, is $\Delta E = E_V - p_V v$ (E_V and p_V are the energy and momentum of a vortex line). A vortex line can be formed if $\Delta E < 0$. As superfluidity disappears when a vortex line appears, the critical velocity v_k is determined by the condition $\Delta E = 0$, i.e., $v_k = E_V / p_V$.

The momentum p_V of a narrow vortex line is given by[4] $p_V = \kappa\rho \int dF_n$, where κ is the circulation of velocity along a contour enclosing the line and ρ is the density and the integration is over a surface bounded by the vortex contour l. In calculating the line energy we shall assume that the vortex line is sufficiently far from the walls for surface effects to be neglected. Then[4]

$$E_v = \frac{\rho}{8\pi} \int \frac{(\operatorname{curl} \mathbf{v}(\mathbf{r}), \operatorname{curl} \mathbf{v}(\mathbf{r}'))}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}\, d\mathbf{r}'.$$

Since p_V is proportional to the square of the linear dimensions of the line and E_V is directly proportional to it, the minimum ΔE corresponds to the maximum line length, coinciding with the transverse dimensions (for a rectangular cross section it is therefore not energetically profitable for circular vortices to be formed instead of rectangular vortices). If, in fact, the line is near the walls, Eq. (1) for E_V is inexact, but it is sufficient for calculating v_k.

According to Feynman,[2] the circulation κ is quantized: $\kappa = 2\pi n_s \hbar/m$, where $n_s = 1, 2, \ldots\ldots$ The smallest values of energy, E_V and of $|\Delta E|$ correspond to $n_s = 1$. By calculating the line energy and momentum we obtain for a circular cross section of radius r

$$v_k = (\hbar / mr)(\ln(r/d) + \ln 16 - 7/4)$$

(d is the diameter of the line cross section, $d \ll r$).

For a rectangular cross section

$$v_k = \frac{\hbar}{m}\left\{\frac{1}{b}\left[\ln\frac{4a(\sqrt{a^2+b^2}-a)}{bd} - \frac{7}{4}\right] + \frac{1}{a}\left[\ln\frac{4b(\sqrt{a^2+b^2}-b)}{ad} - \frac{7}{4}\right] + 2\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}\right\};$$

and for $b \ll a$

$$v_k = (\hbar / mb)\left[\ln(2b/d) + \frac{1}{4}\right].$$

a and b are the sides of the rectangle ($a, b \gg d$). According to Lifshitz and Pitaevskiĭ[5] we should take d as 2×10^{-7} cm. For $r \approx 10^{-5}$ cm we then obtain $v_k \approx 80$ cm/sec. A similar estimate for the flow of helium from a capillary into a beaker was derived by Feynman.

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