NONLINEAR LANGMUIR ELECTRON OSCILLATIONS IN A PLASMA

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An exact solution has been obtained for the nonlinear oscillations of the electron density in a plasma at zero electron temperature. The initial conditions necessary for these oscillations are determined.

1. Langmuir and Tonks,[1] in studying the motion of electrons in a plasma, established the fact that small perturbations lead to harmonic oscillations at a frequency $\omega_0 = (4\pi e^2 n_0/m)^{1/2}$ which are localized in the region of the initial perturbation. However perturbations are not always small. Hence it is of interest to study these oscillations when the magnitude of the perturbation is arbitrary. This problem is considered below.*

2. If we assume that the positive ions of the plasma remain fixed and that the electron temperature is zero,[1,2] in the hydrodynamic approximation the behavior of the plasma is described by the following system of equations:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0, \qquad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{e}{m}\frac{\partial \varphi}{\partial x},$$

$$\partial^2\varphi/\partial x^2 = 4\pi e\,(n - n_0), \quad \mathrm{curl}\,\mathbf{H} = 0, \tag{1}$$

where e, m, v and n are the charge, mass, hydrodynamic velocity, and electron density; φ is the potential of the self-consistent field; n_0 is the positive ion density (in the equilibrium state the plasma may be assumed to be neutral).†

3. The direct solution of the equations in (1) involves considerable mathematical difficulty; however this difficulty can be easily surmounted if the Lagrangian form of the equations of motion is used. By some elementary transformations we obtain the following system from (1):‡

$$n\frac{d^2n}{dt^2} - 2\left(\frac{dn}{dt}\right)^2 + \frac{4\pi e^2}{m}n^2(n - n_0) = 0, \tag{2a}$$

$$\frac{d^2v}{dt^2} + \frac{4\pi e^2 n_0}{m}v = 0. \tag{2b}$$

Introducing the following variables in Eq. (2a):

$$y = \ln(1+\nu), \qquad \nu = (n - n_0)/n_0, \tag{3}$$

we obtain

$$d^2y/dt^2 - (dy/dt)^2 + \omega_0^2(e^y - 1) = 0. \tag{4}$$

The exact solution of Eq. (4) is

$$\nu = b\sin(\omega_0 t + \varphi)/[1 - b\sin(\omega_0 t + \varphi)], \tag{5}$$

where b and φ are constants of integration.* Returning to the variable n we have

$$n = n_0/[1 - b\sin(\omega_0 t + \varphi)]. \tag{6}$$

The law of motion of the point is found from Eq. (2). Taking the origin of coordinates in the plane of symmetry of the perturbation, integrating the equation $dx/dt = v$ we obtain

$$x - x_0 = a(x_0)[1 - \sin(\omega_0 t + \varphi_1)], \tag{7}$$

where $a(x_0)$ is a function which determines the distribution of oscillation amplitudes of the points while φ_1 is the initial phase. If the velocities of the points are zero at $t = 0$, Eqs. (6) and (7) become

$$n = n_0/[1 - b\cos\omega_0 t],$$
$$x - x_0 = a(x_0)[1 - \cos\omega_0 t]. \tag{8}$$

To determine $b(x_0)$ and $a(x_0)$ from the initial distribution $n(x_0)$ we use the relation $n(x_0, t) = dn(x_0)/dx_0$, where $dn(x_0)$ is the number of electrons in the volume element dx_0 at the starting time. As a result we obtain

$$b = \frac{n(x_0) - n_0}{n(x_0)}, \qquad a = \frac{1}{n_0}\int_0^{x_0} n(x_0)\,dx_0 - x_0. \tag{9}$$

*Published studies on nonlinear plasma oscillations refer chiefly to the case of beams or plasmas with initial velocities.[2,3] In a note published by Polovin[4] concerning a fixed plasma it was shown that the frequency of the electron oscillations is independent of the initial perturbations.

†Because of the motion of the electrons the last equation of (1) is valid when the conduction currents are compensated by the displacement currents. This condition is one of the basic properties of longitudinal Langmuir oscillations.

‡In making the transformation use is made of the relation $4\pi env + \partial E/\partial t = 0$; d/dt denotes the total derivative.

*The constant b depends on the initial density perturbation while φ depends on the initial velocity distribution.

4. Equations (8) and (9) give the final solution for nonlinear Langmuir oscillations in Lagrangian form (for an arbitrary initial density distribution but vanishing initial velocities) and gives $n(x, t)$ in parametric form.

By using the Eulerian form we can eliminate x_0 from Eqs. (8) and (9), but this procedure generally means the solution of a transcendental equation. In the case of a linear density distribution at the initial time this elimination process can be carried out relatively easily.

The nonlinear Langmuir oscillations (8) take place at a frequency $\omega_0 = (4\pi e^2 n_0/m)^{1/2}$, which is independent of the initial perturbation.[4] The oscillation amplitudes for individual points (for zero initial velocities) depend only on the initial density distribution. If this distribution is such that $n(x_0)/n_0 \leq \frac{1}{2}$ oscillations are impossible because the corresponding electron density becomes infinite or negative. This situation is a result of the approximation being used. In neglecting the temperature of the electrons we have the eliminated gas-kinetic pressure, which reduces the force tending to compress particles which are initially separated. In the case $n(x_0)/n_0 \leq \frac{1}{2}$ the absence of forces which oppose compression leads to relative particle velocities such that the coordinates of points which are initially separated coincide at later times (generally speaking, for later times $x_1 > x_2$ even if $x_{01} < x_{02}$). The velocities of contiguous points are different. This ambiguity in the velocity field indicates a violation of the conditions required for applying the hydrodynamic approximation.

Taking account of the pressure in the analysis of the nonlinear electron oscillations would lead to an effect similar to that of a discontinuity in a medium with a temperature different from zero.

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[1] L. Tonks and I. Langmuir, Phys. Rev., **33**, 195 (1929).

[2] A. I. Akhiezer and R. V. Polovin, Dokl. Akad. SSSR, **80**, 193 (1951).

[3] H. K. Sen, Phys. Rev., **97**, 849 (1955).

[4] R. V. Polovin, JETP **31**, 354 (1956), Soviet Phys. JETP **4**, 290 (1957).

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