PAIR PRODUCTION IN COLLISIONS BETWEEN CHARGED PARTICLES

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Pair production due to collisions between fast charged particles and atoms is considered. Expressions for the cross sections are obtained which are valid for positron and electron energies comparable with that of the parent particle.

1. INTRODUCTION

THE effect of pair production in collisions between charged particles has been discussed in references 1 to 5. The expressions obtained in these articles for cross sections agree in the region of pair particle energies ϵ_+ and ϵ_- which satisfy the inequality $\epsilon_+ + \epsilon_- \ll E/m$ (region I). The cross sections in the region $\epsilon_+ + \epsilon_- \gg E/m$ (region II) were discussed in the articles by Bhabha[2] and Murota and others.[5] The latter article, written not long ago, shows the error of Bhabha's results in region II. However, it seems that even Murota's work is not exact.

Two types of processes contribute to the cross section for the effect under consideration: (a) processes of the first order, where the pair particles are considered free; (b) processes of the second order, where the parent particle is considered free. Processes of the second order make the main contribution to the integral cross section. However, if we look over the cross sections in region II the contribution of processes of the first order can be decisive.

Finally we note, in reviewing radiative processes in the high-energy region, that the influence of multiple scattering is substantial. Landau and Pomeranchuk[6,7] were the first to draw attention to this fact. The cross sections for bremsstrahlung and pair production by γ quanta were obtained by Migdal[8] taking this effect into consideration. The following article discusses the influence of multiple scattering on pair production by charged particles.

2. CONTRIBUTION OF SECOND-ORDER PROCESSES

We shall examine pair production by singly charged particles with the mass $m \gg 1$ and the energy $E \gg m$.* We also will assume the energy of the electron to be $\epsilon_- \gg 1$ and the energy of the positron to be $\epsilon_+ \gg 1$. The contribution to the differential cross section of second order processes in the lowest approximation theory of excitation gives the expression

$$d\sigma = \frac{e^4}{(k^2 - \omega^2)^2} \overline{|P_\mu G_\mu|^2} \frac{\delta(E - E' - \omega)\, d^3p' d^3p_+ d^3p_-}{(2\pi)^8 J}, \quad (1)$$

where $\mathbf{p}$ is the initial momentum and $\mathbf{p}'$ is the final momentum of the parent particle;

$$\mathbf{k} = \mathbf{p} - \mathbf{p}'; \quad E = \sqrt{p^2 + m^2}, \quad E' = \sqrt{p'^2 + m^2},$$

$$\omega = \varepsilon_+ + \varepsilon_-, \quad \varepsilon_\pm = \sqrt{p_\pm^2 + 1},$$

$$\mathbf{q} = \mathbf{k} - \mathbf{p}_+ - \mathbf{p}_-; \quad J = p/E \approx 1;$$

P_μ is the transition current of the parent particle equal to $(p+p')_\mu/2\sqrt{EE'}$ for the scalar particles and $(\overline{u}_{\mathbf{p}'}\gamma_\mu u_{\mathbf{p}})$ for particles with spin $\frac{1}{2}$;

$$G_\mu = \left(\overline{u}_{\mathbf{p}_-}\left[\gamma_4 \frac{i(\hat{k} - \hat{p}_+) - 1}{(\mathbf{k} - \mathbf{p}_+)^2 + 1}\gamma_\mu + \gamma_\mu \frac{i(\hat{p}_- - \hat{k}) - 1}{(\mathbf{p}_- - \mathbf{k})^2 + 1}\gamma_4\right] v_{\mathbf{p}_+}\right) V_{\mathbf{q}} \quad (2)$$

is the matrix element for pair production by a virtual quantum; the dash indicates an averaging over the initial state spins and the summation over the final state spins.

It is convenient to do the calculation in the coordinate system whose z axis is directed along the vector $\mathbf{n} = \mathbf{k}/k$ (k-system).

Because of gauge invariance in the k-system, the following equations hold:

$$P_3 k + i\omega P_4 = 0, \quad G_3 k + i\omega G_4 = 0,$$

therefore

$$P_\mu G_\mu = P_i G_i + (1 - \omega^2/k^2) P_4 G_4,$$

with the summation for i going from 1 to 2. After integrating over the angles, quantities of the type $\overline{(P_\mu P_\nu^*)}$ with $\mu \neq \nu$ vanish, and $\overline{|P_1|^2}$ and $\overline{|P_2|^2}$ give equal contributions. Therefore we can substitute

*A system of units is used in which $\hbar = m_e = c = 1$.

$$\overline{|P_\mu G_\mu|^2} \to {}^1/_2 \overline{|P_\perp|^2}\,\overline{|G_\perp|^2} + (1-\omega^2/k^2)^2\,\overline{|P_4|^2}\,\overline{|G_4|^2}, \quad (3)$$

where

$$|P_\perp|^2 = |P_1|^2 + |P_2|^2, \quad |G_\perp|^2 = |G_1|^2 + |G_2|^2.$$

The first term in the right side of (3) describes the contribution of transverse photons and the second term describes the contribution of longitudinal and scalar photons.

The calculations of the quantities included in (3) are somewhat more complex than in the theory of pair production by real quanta. We obtain

$$\overline{|G_\perp|^2} = 4\Big[\Big(\frac{1}{M^2+p_+^2\theta_+^2} - \frac{1}{M^2+p_-^2\theta_-^2}\Big)^2 + \frac{p_+^2+p_-^2}{k^2}\Big(\frac{p_+\theta_+}{M^2+p_+^2\theta_+^2} - \frac{p_-\theta_-}{M^2+p_-^2\theta_-^2}\Big)^2\Big]|V_{\mathbf{q}}|^2, \quad (4)$$

which includes notations understandable from the following equations:

$$\mathbf{p}_\pm = \mathbf{n}p_\pm\Big(1-\frac{1}{2}\theta_\pm^2\Big) + p_\pm\boldsymbol{\theta}_\pm, \quad \mathbf{p} = \mathbf{n}p\Big(1-\frac{1}{2}\theta^2\Big) + p\boldsymbol{\theta},$$

$$\mathbf{p}' = \mathbf{n}\Big(p-k-\frac{1}{2}p\theta^2\Big) + p\boldsymbol{\theta},$$

$$M^2 = 1 + p_+p_-(m^2+p^2\theta^2)/p(p-k). \quad (5)$$

The matrix element for pair production by real quanta is obtained from (4) with $M^2 = 1$.

The calculation of $\overline{|G_4|^2}$ is not difficult. With the required accuracy we obtain

$$\overline{|G_4|^2} = 8p_+^2p_-^2k^{-2}[1/(M^2+p_+^2\theta_+^2) - 1/(M^2+p_-^2\theta_-^2)]^2|V_{\mathbf{q}}|^2. \quad (6)$$

For scalar particles

$$|P_\perp|^2 = p^2\theta^2/p(p-k),$$

$$|P_4|^2 = (p-k/2)^2/p(p-k); \quad (7)$$

for particles with spin $\frac{1}{2}$

$$\overline{|P_\perp|^2} = \frac{m^2k^2 + [p^2+(p-k)^2]p^2\theta^2}{2p^2(p-k)^2}, \quad \overline{|P_4|^2} = 1. \quad (8)$$

In addition we find

$$k^2 - \omega^2 = k^2(m^2+p^2\theta^2)/p(p-k). \quad (9)$$

Using (5), the vector $\mathbf{p}'$ can be represented in the form

$$\mathbf{p}' = (p-k)(\mathbf{p}/p - \boldsymbol{\theta}) + p\boldsymbol{\theta} = (\mathbf{p}/p)(p-k) + k\boldsymbol{\theta},$$

whence

$$\delta(E-E'-\omega)\,d^3p'd^3p_+d^3p_- = k^2d\boldsymbol{\theta}p_+^2d\boldsymbol{\theta}_+dp_+p_-^2d\boldsymbol{\theta}_-dp_-. \quad (10)$$

Finally, for the neutral atom

$$V_{\mathbf{q}} = Ze^2/(q^2+q_0^2), \quad (11)$$

where $q_0 = Z^{1/3}/137$ is the inverse Thomas-Fermi radius, and for $\mathbf{q}^2$ we obtain from (5)

$$\mathbf{q}^2 = (p_+\boldsymbol{\theta}_+ + p_-\boldsymbol{\theta}_-)^2 + \tfrac{1}{4}(p_+\theta_+^2 + p_-\theta_-^2 + kM^2/p_+p_-)^2. \quad (12)$$

Now, using (1) to (12) we can obtain the expression for the differential cross section.

Integration over angles must be done next. For this we note that the angles $\theta_\pm < \max\{1/k, m/p\}$ give the main contribution, so that the quantities $p_\pm\theta_\pm^2$ included in q^2 can be ignored. After this we introduce new variables

$$\mathbf{y} = (2M)^{-1}(p_+\boldsymbol{\theta}_+ + p_-\boldsymbol{\theta}_-),$$

$$\mathbf{x} = (2M)^{-1}(p_+\boldsymbol{\theta}_+ - p_-\boldsymbol{\theta}_-), \quad (13)$$

in which integration over $d\mathbf{x}$ is easily done. As a result we obtain

$$\int\overline{|G_\perp|^2}p_+^2d\boldsymbol{\theta}_+p_-^2d\boldsymbol{\theta}_- = 2^6\pi^2Z^2e^4\int_0^\infty\frac{[\psi(y) + M^2(p_+^2+p_+^2)k^{-2}\varphi(y)]\,y\,dy}{[4M^2y^2+(k/2p_+p_-)^2M^4+q_0^2]^2}, \quad (14)$$

$$\int\overline{|G_4|^2}p_+^2d\boldsymbol{\theta}_+p_-^2d\boldsymbol{\theta}_- = 2^7\pi^2Z^2e^4\frac{p_+^2p_-^2}{k^2}\int_0^\infty\frac{\psi(y)\,y\,dy}{[4M^2y^2+(k/2p_+p_-)^2M^4+q_0^2]^2}, \quad (15)$$

where

$$\psi(y) = 1 - \frac{1}{y\sqrt{1+y^2}}\sinh^{-1}y,$$

$$\varphi(y) = \frac{2y^2+1}{y\sqrt{1+y^2}}\sinh^{-1}y - 1. \quad (16)$$

In integrals (14) and (15) the values of y are very small so that they may be calculated approximately with logarithmic accuracy. The lower limit of integration as can be seen from (12) is determined by the largest of the quantities $kM/4p_+p_-$ and $q_0/2M$. In determining the upper limit it is necessary to consider the final size of the nucleus which limits the maximum magnitude of the transmitted momentum to the values $\sim 1/R$ where $R = 0.5\,r_0Z^{1/3}$ is the radius of the nucleus so that $y_{max} \sim 1/RM$. On the other hand, the functions $\psi(y)$ and $\varphi(y)$ significantly increase for values $y \lesssim 1$, therefore the upper limit in any case is not above $y_{max} \sim 1$. Thus we obtain

$$\int |\overline{G_\perp}|^2 p_+^2 d\theta_+ p_-^2 d\theta_- = \frac{8\pi^2 Z^2 e^4}{3M^4}\left[1 + 2M^2 \frac{p_+^2 + p_-^2}{k^2}\right] L,$$

$$\int |\overline{G_4}|^2 p_+^2 d\theta_+ p_-^2 d\theta_- = \frac{16\pi^2 Z^2 e^4}{3M^4} \frac{p_+^2 p_-^2}{k^2} L, \tag{17}$$

where $L = \ln(y_{max}/y_{min})$, with

$$y_{min} = \begin{cases} kM/p_+p_- & \text{for } kM/p_+p_- > q_0/M \\ q_0/M & \text{for } kM/p_+p_- < q_0/M, \end{cases}$$

$$y_{max} = \begin{cases} 1 & \text{for } 2MR < 1 \\ 1/2MR & \text{for } 2MR > 1 \end{cases} \tag{18}$$

It is possible to ignore the dependence of the quantity M on θ inside the logarithm sign without introducing any error, so that in (18) we can assume

$$M = \sqrt{1 + m^2 p_+ p_- / p(p-k)} = \sqrt{1+x}.$$

We also must integrate over $d\theta$. As a result for particles with spin $\frac{1}{2}$

$$d\sigma_{1/2} = \frac{2\alpha^2 Z^2 r_0^2}{3\pi k^2} L \left\{ \frac{p^2 + (p-k)^2}{p^2}\left[A(x) + 2\frac{p_+^2 + p_-^2}{k^2} B(x)\right]\right.$$

$$+ \frac{k^2}{p^2}\left[C(x) + 2\frac{p_+^2+p_-^2}{k^2} D(x)\right]$$

$$\left. + \frac{8p_+p_-}{k^2}\frac{p-k}{p(1+x)}\right\} dp_+ dp_-, \tag{19}$$

and for particles with spin 0

$$d\sigma_0 = \frac{4\alpha^2 Z^2 r_0^2}{3\pi k^2} L \left\{ \frac{p-k}{p}\left[A(x) + 2\frac{p_+^2+p_-^2}{k^2} B(x)\right]\right.$$

$$\left. + \frac{4p_+p_-}{k^2}\frac{(p-k/2)^2}{p^2(1+x)}\right\} dp_+ dp_-. \tag{20}$$

Here

$$x = m^2 p_+ p_- / p(p-k),$$

$$A(x) = (1+2x)\ln\left(1+\frac{1}{x}\right) - 2,$$

$$B(x) = (1+x)\ln\left(1+\frac{1}{x}\right) - 1,$$

$$C(x) = \frac{1+2x}{1+x} - 2x\ln\left(1+\frac{1}{x}\right),$$

$$D(x) = 1 - x\ln\left(1+\frac{1}{x}\right). \tag{21}$$

The first terms in (19) and (20) describe the contribution of transverse photons emitted without spin flip. The last terms are the contribution of longitudinal and scalar photons. The second term in (19) describes the contribution of transverse photons emitted with spin flip of the parent particle.

Now we turn to limiting cases.

I. $k \ll p/m$. Then $x \ll 1$ and according to (19) and (20) we find

$$d\sigma_{1/2} \approx d\sigma_0 \approx \frac{8\alpha^2 Z^2 r_0^2}{3\pi} L \frac{dp_+ dp_-}{k^2}\left[1 + 2\frac{p_+^2+p_-^2}{k^2}\right] \ln\frac{\varkappa p}{km}, \tag{22}$$

where we have replaced $\sqrt{p_+p_-}$ by k inside the logarithm symbol and introduced the magnitude $\varkappa \approx 1$. The result is in complete accord with the results of other authors and can be obtained using the Weiszäcker-Williams method.

IIa. $k \gg p/m$, $x \gg 1$. In this case we obtain from (21) for the functions A, B, C and D

$$2A(x) \approx C(x) \approx 1/3x^2, \quad B(x) \approx D(x) \approx 1/2x. \tag{23}$$

Then

$$d\sigma_{1/2} = \frac{4\alpha^2 Z^2 r_0^2}{3\pi k^2}\frac{p(p-k)}{m^2 p_+ p_-} L\left[\frac{p_+^2+p_-^2}{k^2}\left(1 - \frac{k}{p} + \frac{k^2}{p^2}\right)\right.$$

$$\left. + \frac{4p_+p_-}{k^2}\left(1-\frac{k}{p}\right)\right] dp_+ dp_-, \tag{24}$$

$$d\sigma_0 = \frac{4\alpha^2 Z^2 r_0^2}{3\pi k^2}\frac{p(p-k)}{m^2 p_+ p_-} L\left[\frac{p_+^2+p_-^2}{k^2}\left(1-\frac{k}{p}\right)\right.$$

$$\left. + \frac{4p_+p_-}{k^2}\left(1-\frac{k}{2p}\right)^2\right] dp_+ dp_-. \tag{25}$$

IIb. $k \gg p/m$, $x \ll 1$. In this case we obtain

$$d\sigma_{1/2} = \frac{2\alpha^2 Z^2 r_0^2}{3\pi k^2} L \frac{p^2 + (p-k)^2}{p^2}\left(1 + 2\frac{p_+^2+p_-^2}{k^2}\right)$$

$$\times \ln\frac{p(p-k)}{m^2 p_+ p_-} dp_+ dp_-; \tag{26}$$

$$d\sigma_0 = \frac{4\alpha^2 Z^2 r_0^2}{3\pi k^2} L \frac{p-k}{p}\left(1 + 2\frac{p_+^2+p_-^2}{k^2}\right)$$

$$\times \ln\frac{p(p-k)}{m^2 p_+ p_-} dp_+ dp_-. \tag{27}$$

The latter results, like the general formulas (19) to (21), are different from those obtained by Murota et al.[5] Even though these authors computed by a somewhat different method they obtained an expression for the differential cross section practically the same as ours. However, in doing the integration over angles they made an inadmissible approximation* that led to substantial mistakes when $k > p/m$.

3. CONTRIBUTION OF FIRST-ORDER PROCESSES

The contribution of first order processes can be found using the same method mentioned in Sec. 2. The calculation of the contribution of interference processes is considerably more difficult and therefore we were forced to limit ourselves to a review of the limiting cases where the contribution of processes of either the first or second order can be ignored.

*It was assumed that the substantial values of y are smaller than M/k, which obviously is not true. See (18).

The influence of the external field on the state of the parent particle can be taken into account using a diffraction approximation (see reference 9). Then the contribution of first order processes will be given by a formula analogous to (1), where we must assume

$$\mathbf{P} = \frac{\pi}{\sqrt{EE'}}\left(\frac{2\mathbf{p}-\mathbf{k}}{p'-|\mathbf{p}-\mathbf{k}|} + \frac{2\mathbf{p}'+\mathbf{k}}{p-|\mathbf{p}'+\mathbf{k}|}\right)G,$$

$$P_0 = \frac{\pi}{\sqrt{EE'}}\left(\frac{E+E'}{p'-|\mathbf{p}-\mathbf{k}|} + \frac{E+E'}{p-|\mathbf{p}'+\mathbf{k}|}\right)G,$$

$$G_\mu = (\bar{u}_{p_-}\gamma_\mu v_{p_+}).$$

The quantity G is connected to the scattering matrix

$$\Omega(\rho) = \begin{cases} 0 & \rho < R \\ e^{2i\eta(\rho)} & \rho > R \end{cases}, \quad \eta(\rho) = Z\alpha \ln k\rho$$

by the relation

$$G = \frac{1}{2\pi}\int (1-\Omega(\rho))\, e^{i(\mathbf{p}-\mathbf{k})\boldsymbol{\rho}}\, d\boldsymbol{\rho},$$

where the integration over $d\boldsymbol{\rho}$ is done along the plane perpendicular to the vector $\mathbf{p}'$ and going through the center of the nucleus.

In calculating the quantities $p' - |\mathbf{p}-\mathbf{k}|$ and $p - |\mathbf{p}'+\mathbf{k}|$ it is necessary to use the law of conservation of energy $E' + \omega = E$. In this case it turns out that the magnitudes of $E - E' - k$ and $\omega - k$ can be of the same order of magnitude, and therefore to ignore the second of them in comparison with the first, as done by Rabinovich,[10] is invalid.

All the calculations are done in the same way as in Sec. 2, therefore we only mention the end result. The contribution of first-order processes to the differential cross section, for particles with spin 0, is equal to

$$dw = \frac{8\alpha^2}{\pi}\frac{p-k}{p}\frac{dp_+dp_-}{k^2}\int_0^\infty dy\int_0^\infty q\,dq\,|G(2\widetilde{M}q)|^2$$

$$\times\left[\frac{\widetilde{M}^2}{(1+y)^2}\left(1+\frac{p_+^2+p_-^2}{k^2}y\right)\varphi(q) + \frac{2(p-k/2)^2}{k^2}\psi(q)\right], \tag{28}$$

where $\widetilde{M}^2 = (1+x+y)m^2/x$ and the functions φ and ψ are given by (16).

In integrating over dq in (28) we can approximately assume

$$|G(\xi)|^2 \approx R^2J_1^2(R\xi)/\xi^2 + 4Z^2\alpha^2/(\xi^2+q_0^2)^2.$$

Then the cross section is broken down into the sum of two terms: diffraction dw_d and Coulomb dw_c. In the region $x \ll 1$ both dw_d and dw_c are insignificantly small. But with $x \gg 1$ we find from (28)

$$dw_{c,0} = \frac{8\alpha^2 r_0^2Z^2}{3\pi m^2}\frac{dp_+dp_-}{k^2}\frac{p-k}{p}\frac{p_+^2+p_-^2}{k^2}$$

$$\times\ln\frac{m^2p_+p_-}{p(p-k)}\ln\frac{1}{mRy_{min}}. \tag{29}$$

For particles with spin $\frac{1}{2}$, analogous calculations give

$$dw_{c,1/2} = \frac{4\alpha^2r_0^2Z^2}{3\pi m^2}\frac{dp_+dp_-}{k^2}\frac{p^2+(p-k)^2}{p^2}\frac{p_+^2+p_-^2}{k^2}$$

$$\ln\frac{m^2p_+p_-}{p(p-k)}\ln\frac{1}{mRy_{min}}. \tag{30}$$

In both formulas $y_{min} = \max\{q_0/m,\ km/p(p-k)\}$.

Nuclear interaction makes a more essential contribution for π mesons:

$$dw_{d,0} = \frac{2\alpha^2R^2}{\pi}\frac{p-k}{p}\frac{dp_+dp_-}{k^2}\ln\frac{m^2p_+p_-}{p(p-k)}$$

$$\times\left[\frac{p_+^2+p_-^2}{k^2}\int_0^\infty\frac{\varphi(q)}{q}J_1^2(2mRq)\,dq\right.$$

$$\left.+\frac{2p_+p_-}{k^2}\frac{(p-k/2)^2}{p(p-k)}\int_0^\infty\frac{\psi(q)}{q}J_1^2(2mRq)\,dq\right]. \tag{31}$$

Comparing these results with (24) and (25) we see that the contribution of first-order processes can be ignored for $x \ll m^2$ if the parent particle does not partake in the nuclear interaction. In the opposite case in the whole region of $x \gg 1$ the contributions of first order processes are more substantial.

4. DISCUSSION OF RESULTS

We see thus that the most likely pair production takes place with $x = m^2p_+p_-/p(p-k) \ll 1$. Corresponding expressions for cross sections are given in (22), (26), and (27).

Formula (31) gives the cross section for $x \gg 1$ if the parent particle is nuclear active. If the parent particle is not nuclear active, for $1 \ll x \ll m^2$ the cross section is given by (24), (25), and for $m^2 \ll x$, by (29) and (30).

The theory we have outlined is applicable to electrons in the region $k \ll p$. The region $k \ll p/m$ gives the main contribution to the integral cross section. In agreement with reference 5, independent of the particle spin we find from (22), in the absence of screening,

$$\sigma = (28\alpha^2r_0^2Z^2/27\pi)\ln^3(\varkappa p/m),$$

and with complete screening

$$\sigma = \frac{28\alpha^2r_0^2Z^2}{27\pi}\ln 190Z^{-1/3}\left[3\ln\frac{\varkappa p}{m}\ln\frac{pZ^{1/3}}{190m} + (\ln 190Z^{-1/3})^2\right].$$

In both formulas $\kappa \sim 1$. Corrections to these formulas contain lower orders of the lograrithm of the energy.

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