$\epsilon^{3/2}$ V, and the distance R between them by $\sqrt{\epsilon}$ R.

On the other hand, the energy of interaction of small particles in vacuum is given directly by the formulas of London or Casimir-Polder (at distances which are respectively smaller or larger than the characteristic wavelengths λ_0 in the spectrum), because only the smallness of the dimensions of the interacting systems is used in the derivation of these equations. We shall find it convenient to rewrite these equations in the following form:[3]

$$U_0 = (3\hbar / \pi R^6) \int_0^\infty \alpha^2 (i\xi)\, d\xi, \qquad \text{for} \quad R \ll \lambda_0,$$

$$U_0 = (23\, \hbar c / 4\pi R^7)\, \alpha^2 (0), \qquad \text{for } R \gg \lambda_0, \tag{1}$$

where $\alpha(\omega)$ is the complex polarizability of the particles. Considering that the polarizability of spherical particles of volume V with dielectric constant ϵ_1 is given by

$$\alpha(\omega) = \frac{3}{4\pi} \frac{\varepsilon_1(\omega) - 1}{\varepsilon_1(\omega) + 2} V, \tag{2}$$

and performing the transformation indicated in the beginning of the article, we obtain a final equation for the interaction energy in the liquid

$$U = \frac{27\, \hbar V^2}{16\, \pi^3 R^6} \int_0^\infty \left[\frac{\varepsilon_1 (i\xi) - \varepsilon (i\xi)}{\varepsilon_1 (i\xi) + 2\varepsilon (i\xi)}\right]^2 d\xi, \qquad \text{for } R \ll \lambda_0,$$

$$U = \frac{207}{64\, \pi^3} \frac{V^2}{R^7} \frac{\hbar c}{\sqrt{\varepsilon (0)}} \left[\frac{\varepsilon_1 (0) - \varepsilon (0)}{\varepsilon_1 (0) + 2\varepsilon (0)}\right]^2, \qquad \text{for } R \gg \lambda_0. \tag{3}$$

We note that for the second equation in (3) to be applicable it is enough that the dimensions of the particles be small only compared with the distance between them (and not compared with λ_0).

Equations (3) can be also rewritten in a different form, taking into account the fact that the change in the dielectric constant of a liquid, due to the presence of N particles per unit volume, is equal to

$$\delta\varepsilon = 3NV (\varepsilon_1 - \varepsilon)\, \varepsilon / (\varepsilon_1 + 2\varepsilon). \tag{4}$$

(where $NV \ll 1$; see, for example, reference 3, problems of Sec. 9).

Using (4), we rewrite (3) in the form

$$U = \frac{3\hbar}{16\pi^3 R^6} \int_0^\infty \left(\frac{\partial\varepsilon (i\xi)}{\partial N}\right)^2 \frac{d\xi}{\varepsilon^2 (i\xi)}, \qquad \text{for } R \ll \lambda_0,$$

$$U = \frac{23\, \hbar c}{64\pi^3 R^7 \varepsilon^{5/2} (0)} \left(\frac{\partial\varepsilon (0)}{\partial N}\right)^2, \qquad \text{for } R \gg \lambda_0 \tag{5}$$

[$\epsilon(\omega)$ is the dielectric constant of the mixture and N is the number of particles per unit volume].

We note that in this form Eqs. (5) describe not only the interaction of macroscopic particles suspended in a liquid, but also the interaction of particles with dimensions on the order of interatomic distances, as well as the interaction between molecules of a dissolved substance in a solution. Here, however, the value of the quantity $\partial\epsilon(\omega)/\partial N$ — the derivative of the dielectric constant of the solution with respect to the concentration — can naturally no longer be calculated from Eq. (4), but must be obtained directly from experiment.

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*We use the equations for the absolute temperature zero, and the influence of the temperature on the interaction forces under ordinary conditions is very small.

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A SIMPLE MODEL IN THE THEORY OF SUPERCONDUCTIVITY

Yu. B. RUMER

Institute of Radiophysics and Electronics, Siberian Branch, Academy of Sciences, U.S.S.R.

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THE key to the understanding of superconductivity lies in the Cooper phenomenon, i.e., in the fact that two electrons with opposite momenta and spins near the Fermi surface can form bound states. These states obviously represent bosons, which form a condensate at low temperatures.

In constructing a theory of superconductivity it is then natural to take the Bose condensation of these bosons explicitly into account. In analogy to the theory of superfluidity of Bogolyubov we therefore introduce the boson creation and annihilation operators $c^+(q)$ and $c(q)$.

The Hamiltonian which takes account of the creation and the decay of the bosons is written in the form

$$H = \sum_k \varepsilon(k)\,(a^+(k)\,a(k) + b^+(-k)\,b(-k)) + \sum_q \omega(q)\,c^+(q)\,c(q) + \sum_{k,q} f(q)\,(c^+(q)\,a(k)\,b(q-k) + c(q)\,b^+(q-k)\,a^+(k)), \tag{1}$$

where $a^+(k)$ and $a(k)$ ($b^+(k)$ and $b(k)$) are the Fermion operators for electrons with "right" ("left") spin; $\epsilon(k) = k^2/2m$; $\omega(q)$ is the energy of the bosons; it is important that $\omega(0) \neq 0$, i.e., that the bosons have a non-vanishing rest energy.

We shall further assume that the overwhelming majority of the bosons lies in the condensate. We therefore neglect all terms with $q \neq 0$ in the Hamiltonian (1). This leads to the approximate Hamiltonian

$$H' = \sum_k \varepsilon(k)\,(a^+(k)\,a(k) + b^+(-k)\,b(-k)) + \omega(0)\,c^+(0)\,c(0) + f(0)\sum_k (c^+(0)\,a(k)\,b(-k) + c(0)\,b^+(-k)\,a^+(k)). \tag{2}$$

As in the theory of superfluidity, we disregard the non-commutativity of the operators $c^+(0)$ and $c(0)$, i.e., we regard these operators as numbers. Without loss of generality we may regard

$$c^+(0) = c(0) = \Delta / f(0) \tag{3}$$

as a real number. In this approximation we obtain the model Hamiltonian

$$H_R = \sum_k \varepsilon(k)\,[a^+(k)\,a(k) + b^+(-k)\,b(-k)] + \Delta\sum_k [a(k)\,b(-k) + b^+(-k)\,a^+(k)] + \omega(0)\,\Delta^2/f^2(0). \tag{4}$$

It can be easily verified that the Hamiltonian H commutes with the operator of the total number of electrons:

$$N = \sum_k [a^+(k)\,a(k) + b^+(-k)\,b(-k)] + \sum_q 2c^+(q)\,c(q). \tag{5}$$

Denoting the chemical potential by μ, we see that the operator $H_R - \mu N$ can at once be diagonalized by the canonical transformation of Bogolyubov:

$$a(k) = \xi_k \cos\varphi_k + \eta_k^+ \sin\varphi_k,$$
$$b(-k) = -\xi_k^+ \sin\varphi_k + \eta_k \cos\varphi_k, \tag{6}$$

where

$$\tan 2\varphi_k = \Delta / [\varepsilon(k) - \mu].$$

In terms of the new coordinates ξ_k, η_k we have

$$H_R - \mu N = \sum_k \Big\{[\varepsilon(k) - \mu] - \sqrt{[\varepsilon(k) - \mu]^2 + \Delta^2}\,(1 - \xi_k^+\xi_k - \eta_k^+\eta_k)\Big\} + (\omega(0) - 2\mu)\,\Delta^2 / f^2(0). \tag{7}$$

We average the operator (7) over the state with an energy E corresponding to the given temperature T:

$$\langle H_R - \mu N\rangle_E = \sum_k \{[\varepsilon(k) - \mu] - \sqrt{[\varepsilon(k) - \mu]^2 + \Delta^2}\,(1 - 2n(k))\} + [\omega(0) - 2\mu]\,\Delta^2 / f^2(0), \tag{8}$$

where

$$n(k) = \langle \xi_k^+\xi_k\rangle_E = \langle \eta_k^+\eta_k\rangle_E = [\exp\{\sqrt{[\varepsilon(k) - \mu]^2 + \Delta^2} / T\} + 1]^{-1}.$$

To find Δ we minimize $\langle H_R - \mu N\rangle_E$ with constant $n(k)$. We have

$$\frac{\partial}{\partial\Delta}\langle H_R - \mu N\rangle = -\sum_k \frac{\Delta}{\sqrt{[\varepsilon(k) - \mu]^2 + \Delta^2}} \tanh\frac{\sqrt{[\varepsilon(k) - \mu]^2 + \Delta^2}}{2T} + \frac{2[\omega(0) - 2\mu]}{f^2(0)}\,\Delta = 0, \tag{9}$$

which leads to the following equation for the determination of Δ:

$$1 = \frac{f^2(0)}{2[\omega(0) - 2\mu]} \sum_k \frac{\tanh\{\sqrt{[\varepsilon(k) - \mu]^2 + \Delta^2} / 2T\}}{\sqrt{[\varepsilon(k) - \mu]^2 + \Delta^2}}. \tag{10}$$

Comparing (10) with the corresponding formula of the usual theory, we see that our model leads to superconductivity if $\omega(0) - 2\mu > 0$. At first sight it seems reasonable to set $\omega(0) - 2\mu \approx -2\Delta < 0$, since $\omega(0) - 2\mu$ is just the binding energy of the pair. It should be noted, however, that the Hamiltonian (1) does not take account of the direct interaction of the unpaired electrons with one another. The amplitudes $c^+(0)$ and $c(0)$ and the energy $\omega(0)$ are therefore the renormalized amplitudes and energy of the pair. As our calculations show, we can neglect the four-fermion terms in the Hamiltonian and at the same time renormalize the pair energy such that the inequality $\omega_0 - 2\mu > 0$ is satisfied.

We see that the quantity $f^2(0)/[\omega(0) - 2\mu] = g > 0$ is the same as the interaction constant in the Bardeen-Cooper-Schrieffer theory.

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