EFFECT OF INELASTIC COLLISIONS ON THE VELOCITY DISTRIBUTION OF ELECTRONS

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Submitted to JETP editor March 7, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 490-500 (August, 1959)

The velocity distribution function for electrons in a weakly ionized plasma has been found with account of inelastic collisions. It is shown that the inelastic collisions lead to a sharp drop in the distribution function for electron energies exceeding the excitation (or ionization) energy.

INTRODUCTION

IN finding the velocity distribution function for electrons in inelastic collisions, one usually neglects completely the fact that the function is valid only under the condition that the mean energy of the electrons is much smaller than the excitation (or ionization) energy. In many cases, and in particular, in the study of phenomena taking place in gas discharges, this proves to be a serious limitation. In a number of researches,[1-3] approximate methods of solution have been found which permit the energy losses in inelastic collisions to be taken into account. However, these methods have limited applicability, and apply essentially to cases in which the mean energy of the electrons is not very great, so that the inelastic collisions affect chiefly the tail of the distribution function. Moreover, they assume some particular form for the dependence of the cross section on the velocity; the model of inelastic collisions used by Davydov[3] is very rough, and leads to the divergence of the distribution at small velocities. It should also be noted that there is an error in the work of Smit[2] (see below); however, in the case considered there of small E/p (E is the electric field intensity, p the gas pressure), the error is shown to be unimportant.

Under these conditions, it is desirable to develop a method free from these limitations; the present paper is devoted to such a development.

We consider the case of a spatially homogeneous plasma, in which the ionization is so small that we can neglect the Coulomb interaction. The generalization of this method to the case of a sufficiently strong ionization, in which the electron-electron interaction becomes important, will be considered below.

1. FUNDAMENTAL EQUATIONS

Let us first consider the case in which the electric and magnetic fields do not depend on the time. Then the kinetic equation for the determination of the stationary electron distribution function has the form

$$\{\gamma + [\mathbf{v}\Omega]\}\,\partial f/\partial \mathbf{v} = St_a + \sum_m^M St_e^{(m)} + \sum_n^N St_i^{(n)} + St_r, \quad (1.1)$$

where $\gamma = e\mathbf{E}/m_e$, $\Omega = e\mathbf{H}/m_e c$ while St_a, $St_e^{(m)}$, $St_i^{(n)}$ and St_r are terms taking into account, respectively, the elastic collisions of electrons with molecules, excitation of the m-th level, ionization (n-fold), and recombination (volume for high pressures and surface for low). Here we have neglected collisions of second order, inasmuch as these do not play an important role for weak ionization. Moreover, we confine ourselves to monatomic molecules. In the case of polyatomic molecules, it is necessary to take into consideration the excitation of vibrational and rotational levels, and also dissociation. This can easily be done by an analogous method.

The expression for St_a has the usual form (see, for example, reference 4), while for the inelastic collisions we assume an approximate model and set

$$St_e^{(m)} = -\nu_e^{(m)}(v) f(v, \theta, \varphi) + \nu_e^{(m)}(v_e^{(m)}) \frac{v_e^{(m)}}{v} f(v_e^{(m)}, \theta, \varphi) \quad (1.2)$$

$$St_i^{(n)} = -\nu_i^{(n)}(v) f(v, \theta, \varphi) + \eta\nu_i^{(n)}(v_i^{(n)}) \frac{\tilde{v}_i^{(n)}}{v} f(v_i^{(n)}, \theta, \varphi) + \tilde{\eta}\nu_i^{(n)}(\tilde{v}_i^{(n)}) \frac{\tilde{v}_i^{(n)}}{v} f(\tilde{v}_i^{(n)}, \theta, \varphi), \quad (1.3)$$

where $\nu_e^{(m)} = N_0 v \sigma_e^{(m)}$, $\nu_i^{(n)} = N_0 v \sigma_i^{(n)}$; N_0 is the number of atoms per unit volume; $\sigma_e^{(m)}$ and $\sigma_i^{(n)}$ are the excitation cross sections of the m-th level and the n-fold ionization, averaged over the angles;

$v_e^{(m)} = (v^2 + 2\varepsilon_e^{(m)}/m_e)^{1/2}$, $v_i^{(n)} = (\eta v^2 + 2\varepsilon_i^{(n)}/m_e)^{1/2}$, $\tilde{v}_i^{(n)} = (\tilde{\eta} v^2 + 2\varepsilon_i^{(n)}/m_e)^{1/2}$,

$\epsilon_e^{(m)}$ and $\epsilon_i^{(n)}$ are the corresponding energies of excitation and ionization, while the quantities $1 < \eta < \infty$ and $\tilde{\eta} = \eta/(\eta - 1)$ take into account the energy distribution between the primary scattered and the secondary electrons in the ionization. Inasmuch as the distribution of secondary electrons has a maximum for low velocities, while the scattered distribution has one at high velocities,[5] we can usually assume $\eta \sim 1$.

As regards the term St_r, it is introduced to satisfy the law of conservation of the number of particles. Account of it in the stationary case has importance in principle, although the specific form is shown to be unimportant. We set

$$St_r = -\nu_r(v) f(v, \theta, \varphi), \tag{1.4}$$

where, as follows from (1.1), the relation

$$\int \left(St_r + \sum_{n=1}^{N} St^{(n)}\right) dv = 0 \tag{1.5}$$

must hold. (N is the maximum multiplicity of ionization.) It is easy to prove that if $St_e^{(m)}$ and $St_i^{(n)}$ are written in this manner, the laws of conservation of energy and of the number of particles are satisfied. The law of conservation of momentum is not satisfied. However, inasmuch as the form of the non-isotropic part of the distribution function is fundamentally determined by elastic collisions, this does not have a significant effect on the isotropic part of the distribution function.

Following Davydov,[3] let us expand f in a series of spherical harmonics in velocity space:

$$f(v, \theta, \varphi) = f_0(v) + \mathbf{v}\mathbf{f}_1(v) + \chi(v, \theta, \varphi), \tag{1.6}$$

where $\chi(v, \theta, \varphi)$ represents the remaining terms in the expansion.

If neutral atoms have a Maxwellian distribution with temperature T_0, then it can be shown (see, for example, reference 4) that, accurate to quantities of first order of smallness in $\delta = m_e/m_i$, we obtain

$$St_a(f_0) = \frac{\delta}{v^2}\frac{\partial}{\partial v} v^3 \nu_a(v)\left[f_0 + \frac{kT_0}{m_e v}\frac{\partial f_0}{\partial v}\right], \tag{1.7}$$

$$St_a(\mathbf{v}\mathbf{f}_1) = -\nu_a(v)\mathbf{v}\mathbf{f}_1, \tag{1.8}$$

where ν_a is the frequency of the elastic collisions.

We substitute Eq. (1.6) in Eq. (1.1), and integrate over the angles θ and φ. Taking Eq. (1.7) into account, and also the condition of the orthogonality of the spherical harmonics, we obtain, after a single integration over v:

$$\delta\nu_a(v)\left[f_0 + \frac{kT_0}{m_e v}\frac{\partial f_0}{\partial v}\right] - \frac{1}{3}\gamma\mathbf{f}_1 - \frac{S(f_0)}{v} = 0, \tag{1.9}$$

where

$$S(f) = -\frac{1}{v^2}\left\{\sum_m^M \int_v^{v_e^{(m)}} \nu_e^{(m)}(v) f(v) v^2 dv - \int_0^v \nu_r(v) f(v) v^2 dv + \sum_n^N \left[\int_v^{v_i^{(n)}} \nu_i^{(n)}(v) f(v) v^2 dv + \int_0^{\tilde{v}_i^{(n)}} \nu_i^{(n)}(v) f(v) v^2 dv\right]\right\}. \tag{1.10}$$

Multiplying (1.1) by $\mathbf{v}$, and again integrating over the angles, we find (neglecting* the correction χ in comparison with f_0):

$$\nu_a(v)\mathbf{f}_1 + [\Omega \times \mathbf{f}_1] + \frac{1}{v^3}\frac{\partial}{\partial v} v^2 S(v\mathbf{f}_1) = -\frac{\gamma}{v}\frac{\partial f_0}{\partial v}. \tag{1.11}$$

Since, in writing down the initial equation, we have neglected the law of conservation of momentum for inelastic collisions, we can also neglect the third term on the left in Eq. (1.11) in comparison with the first. Then, solving (1.11) relative to $\mathbf{f}_1$, and substituting the resultant expression in (1.9), we obtain an equation for the isotropic part of the distribution function $f_0(v)$:

$$\delta\nu_a(v)\left[f_0 + \frac{kT_0}{m_e v}\frac{\partial f_0}{\partial v}\right] + \frac{\alpha\gamma}{3\nu_a}\frac{1}{v}\frac{\partial f_0}{\partial v} - \frac{S(f_0)}{v} = 0, \tag{1.12}$$

where

$$\alpha(v) = \frac{\gamma + [\gamma \times \Omega]/\nu_a + \Omega(\gamma\Omega)/\nu_a^2}{1 + \Omega^2/\nu_a^2}. \tag{1.13}$$

Inasmuch as the magnetic field enters into Eq. (1.12) only in the form of the product $\alpha \cdot \gamma$, we shall everywhere below assume for simplicity that $\Omega = 0$, having in view that the generalization of results to the case $\Omega \neq 0$ reduces simply to replacing the quantity $\mu(u)$ in the obtained formulas (see 1.18) by

$$\mu_H(u) = \frac{3kT_0\,\delta\tilde{\nu}_a^2}{m_e\gamma^2}\varphi_a^2(u) + \frac{\tilde{\nu}_a^2\varphi_a^2(u) + \Omega^2\cos^2(\widehat{\gamma\Omega})}{\tilde{\nu}_a^2\varphi_a^2(u) + \Omega^2}. \tag{1.13a}$$

We change in (1.12) to a new independent variable u and to a new function A(u) by the formulas

$$u = m_e v^2/2\varepsilon_e^{(1)}, \quad f_0(v)|_{v=v(u)} = f_{00}(u) A(u), \tag{1.14}$$

where $f_{00}(u)$ is the solution of the equation in the absence of the inelastic collisions. Now, setting

$$\nu_a = \tilde{\nu}_a \varphi_a(u), \quad \nu_r = \tilde{\nu}_r \varphi_r(u) u^{-1/2},$$
$$\nu_e^{(m)} = \tilde{\nu}_e^{(m)} \varphi_e^{(m)}(u) u^{-1/2}, \quad \nu_i^{(n)} = \tilde{\nu}_i^{(n)} \varphi_i^{(n)}(u) u^{-1/2}, \tag{1.15}$$

where $\tilde{\nu}_a$, $\tilde{\nu}_r$, $\tilde{\nu}_e^{(m)}$ and $\tilde{\nu}_i^{(n)}$ are certain constants, chosen to make $\varphi_a(1) = \varphi_r(1) = 1$ and to make the functions $\varphi_e^{(m)}$ and $\varphi_i^{(n)}$ of the order of unity for $u \gg 1$, we get

$$\frac{dA}{du} + Q(u)\left[F\{A\} - \lambda_r^2 \int_0^u \varphi_r f_{00} A\, du\right] = 0. \tag{1.16}$$

*It can be shown that for $\lambda_e^2 \gg 1$, the function $\chi \sim (f_0\tilde{\nu}_e/\tilde{\nu}_a)\lambda_e^{-(2q+1)/(q+1)} \ll f_0$.

Here

$$F\{A\} = \sum_m^M \lambda_{e,m}^2 \int_u^{u+u_e^{(m)}} f_{00}(u)\,\varphi_e^{(m)}(u)\,A(u)\,du + \sum_n^N \lambda_{in}^2 \Big[\int_u^{\eta u+u_i^{(n)}} f_{00}(u)\,\varphi_i^{(n)}(u)\,A(u)\,du + \int_0^{\tilde{\eta} u+u_i^{(n)}} f_{00}(u)\,\varphi_i^{(n)}(u)\,A(u)\,du \Big], \quad (1.17)$$

$$Q(u) = \varphi_a(u)/u^{3/2} f_{00}\mu, \quad \mu = (3kT_0\,\delta\tilde{\nu}_a^2/m_e\gamma^2)\varphi_a^2(u) + 1,$$

$$f_{00} = \exp\Big\{-\int_0^u du/u_T\Big\}, \quad u_T = \gamma^2 m_e|\mu(u)/3\delta\tilde{\nu}_a^2\varepsilon_e^{(1)}\varphi_a^2(u), \quad (1.18)$$

$$\lambda_{em}^2 = 3\varepsilon_e^{(1)}\tilde{\nu}_a\tilde{\nu}_e^{(m)}/2m_e\gamma^2, \quad \lambda_{in}^2 = 3\varepsilon_e^{(1)}\tilde{\nu}_a\tilde{\nu}_i^{(n)}/2m_e\gamma^2 = \lambda_{em}^2\beta_{nm}^2,$$

$$\lambda_r^2 = 3\varepsilon_e^{(1)}\tilde{\nu}_a\tilde{\nu}_r/2m_e\gamma^2,$$

$$u_e^{(m)} = \varepsilon_e^{(m)}/\varepsilon_e^{(1)} \geqslant 1, \quad u_i^{(n)} = \varepsilon_i^{(n)}/\varepsilon_e^{(1)} > 1.$$

We note that, in accord with (1.5),

$$\lambda_r^2 \int_0^\infty \varphi_r f_{00} A\,du = \sum_n^N \lambda_{in}^2 \int_0^\infty \varphi_i^{(n)} f_{00} A\,du. \quad (1.19)$$

This condition is very important. Actually, if it is not satisfied, then, as follows from (1.16), A(u) vanishes for a finite value of u, whereas its derivative at this point undergoes a jump, which is physically without meaning. In essence, (1.19) is equivalent to the requirement

$$\partial A/\partial u \to 0 \quad \text{for } u \to \infty, \quad A(u) \to 0. \quad (1.20)$$

In the work of Smit,[2] the loss of particles as the result of recombination was not taken into account; hence condition (1.19) was not fulfilled. However, in the solution of the equation, the author used the condition (1.20) (which is not compatible with the initial equation in this case), and thus obtained correct results for u > 1.

2. SOLUTION OF THE EQUATION FOR A(u)

To avoid needless complications, we first consider the case in which only a single ionization (N = 1) is possible, and there is either only one level of excitation (M = 1), or the distance between the levels is so small ($u_e^{(M)} - 1 \ll 1$) that $\epsilon_e^{(1)}$ can be replaced by some average excitation energy $\bar{\epsilon}_e$, while $\varphi_e^{(m)}$ can be replaced by the mean frequency of excitation.

An exact solution of Eq. (1.16) is evidently impossible without going to numerical integration. However, one can attempt to apply an approximate method of solution, which is essentially the expansion of A(u) in inverse powers of the parameter λ_{e1}^2. Actually, as follows from its definition, this parameter is of the same order of magnitude as the ratio of the energy lost per unit time by the electron (with velocity corresponding to the maximum of the frequency of ionization) on excitation to the energy obtained by it from the external field, i.e., it is usually much greater than unity.* Thus, we shall solve the equation for A(u) under the assumption that $\lambda_{e1}^2 \gg 1$.

We note that, as follows from the discussion below, the function A(u) falls off with distance as $\sim 1/\lambda_{e1}$, whereas $f_{00}(u)$ depends on distance as $\sim 1/\delta\lambda_{e1}^2$. Consequently, since $\delta \ll 1$, we can, for not very large λ_{e1} (less than δ^{-1}), remove the function f_{00} in Eqs. (1.16), (1.17) from under the integral sign.† Then, denoting by $A_I(u)$ the solution for $u \leq 1$, and by $A_{II}(u)$ the solution for $u \geq 1$, and dropping the index (1) in λ_{e1}, ϵ_{11}, $\varphi_e^{(1)}$ and $\varphi_i^{(1)}$, we obtain

$$A_{II}'(u) + Q_1(u)\Big[\lambda_e^2 F_1\{A_{II}\} - \lambda_r^2 \int_0^u \varphi_r A_{II}\,du\Big] = 0 \quad (2.1)$$

for $u \geq 1$ and

$$A_I'(u) - \lambda_r^2 Q_1(u) \int_0^u \varphi_r A_I\,du = -\lambda_e^2 Q_1(u) F_1\{A_{II}\} \quad (2.2)$$

for $u \leq 1$, where

$$F_1\{A_{II}\} = \int_u^{u+1} \varphi_e(u)\,A_{II}(u)\,du + \beta^2\Big[\int_u^{\eta u+u_i} \varphi_i(u)\,A_{II}(u)\,du + \int_0^{\tilde{\eta} u+u_i} \varphi_i(u)\,A_{II}(u)\,du\Big], \quad (2.3)$$

$$Q_1(u) = \varphi_a(u)/u^{3/2}\mu(u). \quad (2.4)$$

The prime here and below denotes differentiation with respect to the argument.

To begin, let us consider Eq. (2.1). We assume that the condition

$$[\ln A_{II}(u)]' \gg 1, \quad (2.5)$$

is satisfied, i.e., the function $A_{II}(u)$ falls off rapidly in the interval [u, u+1], and differentiate (2.1) with respect to u. Then, considering that, by (1.19), $\lambda_r^2 \lesssim 1$ and $\lambda_e^2 \gg 1$, and neglecting $A_{II}(u+1)$ in comparison with $A_{II}(u)$, we find, with accuracy $\gtrsim \lambda_e^{-2}$, the equation of first approximation

*The case of very large E/p, when the quantity λ_{e1}^2 becomes smaller than unity, will not now be considered, inasmuch as for such values of E/p, the percent ionization is sufficiently high and it is necessary to take into account the electron-electron interaction.

†The case $\lambda_{e1} > \delta^{-1}$ always corresponds to low (of the order of several hundred degrees) mean energies of electrons, when the inelastic collisions no longer play a role.

$$(A_{II}^{(1)})'' - (\ln Q_1(u))'(A_{II}^{(1)})' - \lambda_e^2 p^2(u) A_{II}^{(1)} = 0, \quad (2.6)$$

where

$$p^2(u) = Q_1(u)[\varphi_e(u) + \beta^2\varphi_i(u)]. \quad (2.7)$$

Up to now we have made no assumptions concerning the form of the dependence of the cross section on the velocity. We now assume that φ_e, φ_i, and φ_a are such that p^2 vanishes nowhere, except at the point $u = 1$, while as $u \to 1$,

$$p^2(u) = p_0^2(u-1)^{2q}, \quad (2.8)$$

where $q > 0$ and $p_0^2 > 0$ are certain constants which depend on the behavior of the functions φ_e and φ_i close to the thresholds of excitation and ionization.

Let us change variables in Eq. (2.6), introducing the independent variable

$$x(u) = \left[(q+1)\lambda_e \int_1^u p(u)\,du\right]^{1/(1+q)} \quad (2.9)$$

and the new function

$$Y(x) = (x'(u)/Q_1(u))^{1/2} A_{II}^{(1)}(u)|_{u(x)}. \quad (2.10)$$

We then get for $Y(x)$ the equation

$$Y''(x) - [\zeta(x) + x^{2q}]Y(x) = 0, \quad (2.11)$$

where

$$\zeta(x) = \tfrac{1}{2}\{(x')^{-1/2}[(x')^{-1/2}x'']' + (x'Q_1)^{-2}[\tfrac{3}{2}(Q_1')^2 - Q_1Q_1'']\}|_{u(x)}.$$

An estimate shows that the function $\zeta(x)$ will, generally speaking, be of the order of $2^{-1/q}(\lambda_e p_0)^{-1/q(q+1)}$, i.e., very small both for x close to zero and for large values of x. It can become large only if u approaches the next root of the function $p^2(u)$. Inasmuch as we have assumed that the function $p^2(u)$ never vanishes for $u > 1$, we can neglect $\zeta(x)$ in Eq. (2.11) in comparison with x^{2q} and obtain

$$Y'' - x^{2q}Y = 0, \quad (2.12)$$

whence, by taking account of the condition (1.20),

$$Y = \text{const}\cdot C_0 K_{1/2(q+1)}\left(\frac{x^{q+1}}{q+1}\right)x^{1/2} = \text{const}\cdot V_q(x), \quad (2.13)$$

where K_n is the MacDonald function of order n and C_0 is a constant, so chosen that $V_q(0) = 1$. Thus

$$A_{II}^{(1)}(u) = \text{const}\cdot[Q_1/x']^{1/2} V_q(x)|_{x(u)}, \quad (2.14)$$

and the condition (2.5) takes the form

$$\lambda_e^2 p^2(u) \gg 1. \quad (2.15)$$

It then follows that, for sufficiently large u, when $\lambda_e^2 p^2(u) \sim 1$, the solution (2.14) turns out to be invalid. However, for these values of u, the function $A_{II}^{(1)}$ is already very small ($\sim \exp[-\lambda_e^2 p_0]$) and consequently is not of interest.

For $\lambda_e^2 p_0^2 \gg 1$, we can limit ourselves entirely to the first approximation $A_{II}^{(1)}$. If $\lambda_e p_0$ becomes of the order of unity, then it is necessary to use the solution $A_{II}^{(2)}$, obtained in second approximation. Assuming, as before, that for $u > 1$,

$$\lambda_i^2 \int_0^{\eta u + u_i} \varphi_i A\,du \approx \lambda_r^2 \int_0^u \varphi_r A\,du,$$

we find from (2.1) that

$$A_{II}^{(2)}(u) = \lambda_e^2 \int_u^\infty Q_1(u) J\{A_{II}^{(1)}\}\,du, \quad (2.16)$$

where

$$J\{A_{II}^{(1)}\} = \int_u^{u+1} \varphi_e A_{II}^{(1)}\,du + \beta^2 \int_u^{\tilde{\eta}u + u_i} \varphi_i A_{II}^{(1)}\,du. \quad (2.17)$$

We now proceed to the finding of the function $A_I(u)$. Generally speaking, for an exact solution of Eq. (2.2) it is necessary to give an explicit form for $\varphi_r(u)$, which, unfortunately, is very little known. We can prove, however, that in the case under discussion here, the term proportional to λ_r^2 plays an important role only in the region of small values of u, where it leads to a certain increase in $A_I(u)$. Moreover, inasmuch as the distribution function for $u < 1$ is needed essentially only for the calculation of the normalizing factor, then, taking it into account that $\tilde{\eta} \gg \eta$, we can set, without excessive error,

$$\lambda_e^2 F_1\{A_{II}^{(1)}\} - \lambda_r^2 \int_0^u \varphi_r A_I\,du = \lambda_e^2 J\{A_{II}^{(1)}\},$$

whence

$$A_I(u) = A_{II}^{(2)}(1) + \lambda_e^2 \int_u^1 Q_1(u) J\{A_{II}^{(1)}\}\,du$$
$$= \lambda_e^2 \int_u^\infty Q_1(u) J\{A_{II}^{(1)}\}\,du. \quad (2.18)$$

The results obtained for $A_I(u)$ and $A_{II}(u)$ can be materially simplified by making use of the fact that $V_q(x)$ is a rapidly decreasing function. We first note that: a) if u_i is close to unity, so that $\alpha(u_i - 1) < 1$, where $\alpha = (\lambda_e p_0)\, e^{1/(q+1)}$, then we can write

$$J = [1+\beta^2]^{-1}\left\{\int_u^{u+1} Q_1^{-1}p^2 A_{II}^{(1)}\,du + \beta^2 \int_u^{\tilde{\eta}u+1} Q_1^{-1}p^2 A_{II}^{(1)}\,du\right\}; \quad (2.19)$$

b) in the opposite case, when $\alpha(u_i - 1) > 1$, we can neglect the second component in (2.17) and get

$$J = \int_u^{u+1} Q_1^{-1}p^2 A_{II}^{(1)}\,du, \quad (2.19')$$

i.e., case b) is obtained from a) if we set $\beta^2 = 0$ in the latter. Therefore, we limit our considerations to the case a).* Substituting (2.14) in Eqs. (2.16) and (2.18), and integrating, we obtain [bearing in mind that, for $u > 1$, the function $A_{II}^{(1)}(u)$ decreases rapidly]:

$$A_{II}(u) = \text{const} \cdot \left[\frac{\alpha Q_1(u)}{x' Q_1(1)}\right]^{1/2} \Big\{ V_q(x(u)) - (1+\beta^2)^{-1} \Big[\frac{x'(u)}{x'(u+1)} V_q(x(u+1)) + \frac{\beta^2}{\tilde{\eta}} \frac{x'(u)}{x'(\tilde{\eta}u+1)} V_q(x(\tilde{\eta}u+1)) \Big] \Big\}, \quad (2.20)$$

where the index (2) in $A_{II}^{(2)}(u)$ is omitted, and

$$A_I(u) = A_{II}(1) - (1+\beta^2)^{-1}\alpha \Big\{ \tau(u) [(1+\beta^2) V_q'(0) - V_q'(x(u+1)) - \beta^2 V_q'(x(\tilde{\eta}u+1))] - x'(u+1) \int_{u+1}^{2} \tau(u-1) x^{2q} V_q(x(u))\, du - \beta^2 x'(\tilde{\eta}u+1) \int_{\tilde{\eta}u+1}^{\tilde{\eta}+1} \tau\left(\frac{u-1}{\tilde{\eta}}\right) x^{2q} V_q(x(u))\, du \Big\}, \quad (2.21)$$

where

$$\tau(u) = \int_u^1 \frac{Q_1(u)}{Q_1(1)}\, du. \quad (2.22)$$

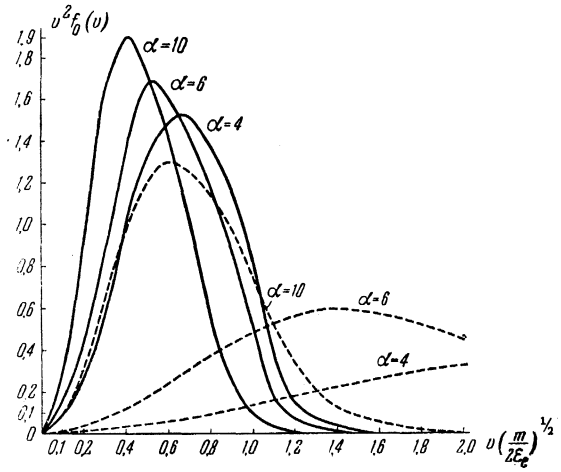
The resulting values apply to the case in which only single ionization is possible ($N = 1$) and there is only one excitation level ($M = 1$). However, it is easy to generalize it to the case of arbitrary N and M. Actually, Eqs. (2.9) and (2.14) for $A_{II}^{(1)}$ and Eqs. (2.16) and (2.18) for $A_I(u)$ and $A_{II}^{(2)}(u)$ evidently remain valid even in the case in which we replace the quantity $\lambda_e^2 p^2(u)$ appearing in it by

$$P^2(u) = Q_1(u) \Big\{ \sum_m^M \lambda_{em}^2 \varphi_e^{(m)} + \sum_n^N \lambda_{in}^2 \varphi_i^{(n)} \Big\}, \quad (2.7')$$

and the expression $\lambda_e^2 J\{A_{II}^{(1)}\}$ by

$$I\{A_{II}^{(1)}\} = \sum_m^M \lambda_{em}^2 \int_u^{u+u_e^{(m)}} \varphi_e^{(m)} A_{II}^{(1)}\, du + \sum_n^N \lambda_{in}^2 \int_u^{\tilde{\eta}u+u_i^{(n)}} \varphi_i^{(n)} A_{II}^{(1)}\, du. \quad (2.17')$$

It follows from the expression for $A_{II}(u)$ that the presence of inelastic collisions leads to a sharp decrease in the distribution function when $u > 1$. Thus, with increase in the intensity of the electric field, the mean energy of the electrons at first increases rapidly and then, beginning with some value $E = E(N_0)$ (corresponding to $\lambda_e^2 \delta \sim 1$), it remains practically constant until values are reached, for which the energy drawn from the field per unit time is comparable with the losses due to excitation and ionization ($\lambda_e^2 \sim 1$), after which it again begins to increase.



The formulas given above are valid, in general, for all $q > 0$. However, the greatest interest attaches to the case $q = 1/2$, inasmuch as the function V_q coincides in this case (with accuracy up to some constant) with the excellently tabulated Airy functions,[6] which simplifies appreciably the use of the formulas. In the same way, the case $q = 1/2$ agrees well with the actual course of the cross section close to the threshold for excitation and ionization.[7]

By way of illustration we have plotted the function $v^2 f_0(v)$ for the case in which the frequency of elastic collisions does not depend on the velocity:

$$\varphi_a = 1, \quad \varphi_e = u(u-1)/(u^2+u_0^2), \quad \tilde{\nu}_e = 2u_0\tilde{\nu}_a, \quad \delta = 5\cdot 10^{-4}, \quad u_0 = 5, \ \beta^2 = 0, \ T = 300^\circ\,\text{K}.$$

The dashed curves correspond to distribution functions computed without consideration of inelastic collisions, the solid curves, with consideration of losses by excitation.

In conclusion, it should be noted that although the method of solution used here is valid, strictly speaking, only for the condition (2.15), an analysis shows that the expressions obtained for $A(u)$ remain valid up to values $\alpha \sim 1$, which corresponds to quite large values of E/p.

3. DISTRIBUTION FUNCTION IN AN ALTERNATING ELECTRIC FIELD

In previous section we found an expression for the electron distribution function in electric and magnetic fields that were constant in time. The results are easily generalized to the case of an alternating electric field. We assume, for sim-

*Similar simplifications can be carried out also in the case of an intermediate value of u_i. However, inasmuch as the resulting formulas are very cumbersome, we shall not write them out.

plicity, that $\mathbf{E}(t) = \mathbf{E}_0 \cos \omega t$. We then have in place of (1.9) and (1.11):

$$\frac{\partial f_0}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v} v^3 \left\{ \delta\nu_a \left[f_0 + \frac{kT_0}{m_e v}\frac{\partial f_0}{\partial v} \right] - \frac{\gamma(t)\,\mathbf{f}_1}{3} - \frac{S(f_0)}{v} \right\}, \quad (3.1)$$

$$\frac{\partial \mathbf{f}_1}{\partial t} + \nu_a \mathbf{f}_1 + [\Omega \times \mathbf{f}_1] = -\frac{\gamma(t)}{v}\frac{\partial f_0}{\partial v}, \quad (3.2)$$

where $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}_0 \cos \omega t$, and $S(f)$ is determined by (1.10) as before.

Before undertaking the solution of these equations, we note that in a variable field the mean energy of the electrons changes only as a result of a change in the external electric field. Therefore, depending on the ratio between the characteristic time $\tau_E \sim 1/\omega$ of change in the field and the time τ of the relaxation of the energy of the electrons, we can segregate the cases of slowly ($\omega\tau \ll 1$) and rapidly ($\omega\tau \gg 1$) changing fields. In the first case (quasi-stationary), the energy distribution is established in a time much shorter than $1/\omega$ and thus the stationary solution is valid provided we replace γ^2 in it by $\gamma^2(t)$. In the second case the mean energy of the electrons does not succeed in changing within the time $\sim 1/\omega$ and, consequently, the isotropic part of the distribution function $f_0(v)$ should not, in first approximation, depend on the time. Consequently, we can solve the set of equations (3.1) and (3.2) by successive approximation, expanding f_0 and $\mathbf{f}_1$ in positive (for $\omega\tau \ll 1$) or negative (for $\omega\tau \gg 1$) powers of the parameter $\omega\tau$. A logical exposition of this method, for the case in which we can neglect the inelastic collisions [$S(f) = 0$], has been given by A. V. Gurevich.[8] In the presence of inelastic collisions, the problem is somewhat complicated, inasmuch as for $S(f) \neq 0$ it is no longer possible to introduce a single parameter independent of the value of the applied electric field capable of characterizing the rate of establishment of the symmetric part of the distribution function. Actually, in weak fields, when the inelastic collisions do not play an important role, the relaxation time is determined essentially by the elastic interaction and $\tau \sim 1/\delta\nu_a$. With increasing electric field, the frequency of the inelastic collisions increases, and the inelastic interaction begins to play a decisive role, i.e., the relaxation time decreases. However, in spite of this indeterminacy in the choice of τ, we can nevertheless evaluate the lower (τ_1) and upper (τ_2) bounds of τ; thus, for frequencies $\omega \ll \tau_2^{-1}$ and $\omega \gg 1/\tau_1$, we apply the method of successive approximations as before, expanding f_0 and $\mathbf{f}_1$ in powers of the parameter $\omega\tau_2$ (for $\omega\tau_2 \ll 1$) or $(\omega\tau_1)^{-1}$ (for $\omega\tau_1 \gg 1$). Solution of these equations in the case of intermediate frequencies $1/\tau_1 > \omega > 1/\tau_2$ entails great mathematical difficulty and will not be considered here.

1. Let $\omega\tau_2 \ll 1$. Then the equations for the functions of first approximation, $f_0^{(1)}$ and $\mathbf{f}_1^{(1)}$, are obtained from (3.1) and (3.2) by discarding the time derivatives, after which we get for $f_0^{(1)}$:

$$\delta\nu_a \left[f_0^{(1)} + \frac{kT_0}{mv}\frac{\partial f_0^{(1)}}{\partial v} \right] + \frac{\alpha(t)\,\gamma(t)}{3\nu_a v}\frac{\partial f_0^{(1)}}{\partial v} - \frac{S(f_0^{(1)})}{v} = 0,$$

where

$$\alpha(t) = \frac{\gamma(t) + [\gamma(t)\times\Omega]/\nu_a + \Omega(\Omega\gamma(t))/\nu_a^2}{1 + \Omega^2/\nu_a^2}.$$

It is easy to see that this equation is identical with Eq. (1.12), the solution of which was obtained above. An estimate of next higher approximations shows that the parameter τ_2 is equal, in order of magnitude, to $1/\delta\bar{\nu}_a$, where $\bar{\nu}_a$ is the average frequency of elastic collisions of the electron with the atoms of the gas.

Thus, for $\omega/\delta\bar{\nu}_a \ll 1$,

$$f_0^{(1)}(v, t) = C(t)\, f_{00}(u, t)\, A(u, t) \quad (3.3)$$

[with $u = u(v)$], where the function $C(t)$ is determined from the normalization condition, and the expressions for $f_{00}(u, t)$ and $A(u, t)$ are given by Eqs. (1.18), (2.16), and (2.18), in which the quantity γ^2 has been replaced by $\gamma^2(t) = \gamma_0^2 \cos^2 \omega t$.

2. For $\omega\tau_1 \gg 1$, the equations of first approximation give

$$\frac{\partial f_0^{(1)}}{\partial t} = 0, \quad \mathbf{f}_1^{(1)} = -\frac{1}{v}\frac{\partial f_0^{(1)}}{\partial v} \operatorname{Re}\left\{ \frac{\gamma_0(\nu_a + i\omega)^2 + (\nu_a + i\omega)[\gamma_0\times\Omega] + \Omega(\Omega\gamma_0)}{(\nu_a + i\omega)[(\nu_a + i\omega)^2 + \Omega^2]} e^{i\omega t} \right\}. \quad (3.4)$$

Proceeding to the second approximation, $f_0^{(2)}$, and bearing in mind that $f_0^{(2)}$ is bounded as $t \to \infty$, we find an equation for $f_0^{(1)}$ as a function of v:

$$\delta\nu_a \left[f_0^{(1)} + \frac{kT_0}{mv}\frac{\partial f_0^{(1)}}{\partial v} \right] + \frac{\gamma_0^2 \bar{\psi}}{6\nu_a v}\frac{\partial f_0^{(1)}}{\partial v} - \frac{S(f_0^{(1)})}{v} = 0, \quad (3.5)$$

where

$$\bar{\psi} = (1 + \omega^2/\nu_a^2)^{-1} \left\{ 1 - \Omega^2 \frac{(\nu_a^2 + \Omega^2 - 3\omega^2)\sin^2(\widehat{\gamma_0, \Omega})}{(\nu_a^2 + \omega^2 + \Omega^2) - 4\omega^2\Omega^2} \right\}. \quad (3.6)$$

Comparing (3.5) with (1.12), we find that they coincide if $\gamma \cdot \alpha$ in (1.12) is replaced by $\gamma_0^2\bar{\psi}/2$. Thus, as has already been pointed out, the isotropic part of the distribution function does not depend in first approximation on the time, and is equal to

$$f_0^{(1)}(v) = \text{const} \cdot f_{00}(u)\, A(u), \quad (3.7)$$

where $f_{00}(u)$ and $A(u)$ are determined, as above, by Eqs. (1.18), (2.16), and (2.18), in which γ^2 is

replaced by $\gamma_0^2\overline{\psi}/2$ and the function $\mu(u)$ [see Eq. (1.18)] is replaced by

$$\widetilde{\mu}(u) = (6kT_0\delta\widetilde{\nu}_a^2/\gamma_0^2 m)\varphi_a^2 + \overline{\psi}(v), \qquad v = v(u). \quad (3.8)$$

By computing the higher approximations, we find that the solution (3.7) is valid only for frequencies ω satisfying the inequality

$$\omega \gg \widetilde{\nu}_e \lambda_e^{-(2q+1)/(q+1)}. \quad (3.9)$$

The author thanks M. S. Rabinovich for his interest in the research.

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Translated by R. T. Beyer
88