ON TAKING THE COULOMB EFFECTS INTO ACCOUNT IN THE THEORY OF SUPERCONDUCTIVITY*

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Through a transition from the energy operator to the S matrix the compensation equation for dangerous electron diagrams is brought into a symmetric form, which is expressed in terms of the usual Green functions. In the high-density electron gas approximation the summation of the Coulomb singularities in the kernel of the compensation equation is performed by the renormalization group method. In the lowest approximation the result of the summation is the same as the formulas obtained previously by qualitative means.

1. THE SYMMETRIC COMPENSATION EQUATION

THE compensation equation for the dangerous electron diagrams in the theory of superconductivity can [according to Eq. (5.19) of the book by Bogolyubov, Tolmachev, and the author[1]] be written in the form

$$\int_{-\infty}^{0} \left\langle \frac{\delta^2 R}{\delta\alpha^+_{k1}(t)\,\delta\alpha^+_{k0}(t')} \right\rangle_c e^{i\tilde{\varepsilon}(k)(t+t')}dtdt' = 0, \qquad (1.1)$$

where R is the energy operator

$$R = H_{int}T\left(\exp\left\{-i\int_{-\infty}^{0} H_{int}(t)\,dt\right\}\right) = H_{int}S^0_{-\infty},$$

$\alpha^+_{k_1}$, $\alpha^+_{k_0}$ are the creation operators for quasi-electrons, $\tilde{\epsilon}(k)$ the energy of the single fermion excitations relative to the Fermi surface, and the index "c" indicates an average over the strongly connected diagrams (see reference 1).

In a previous paper[2] we formulated a rule for going over from matrix elements of the variational derivatives of the energy operator R to the matrix elements of the variational derivatives of the Feynman matrix

$$S = S^{\infty}_{-\infty} = T\left(\exp\left[-i\int_{-\infty}^{\infty} H_{int}(t)\,dt\right]\right).$$

According to Eq. (2.9) of that paper, Eq. (1.1) can be replaced for small $\tilde{\epsilon}(k)$ (i.e., the immediate vicinity of the Fermi surface) by

$$i\int_{-\infty}^{\infty} \left\langle \frac{\delta^2 S}{\delta\alpha^+_{k1}(0)\,\delta\alpha^+_{k0}(t)} \right\rangle_c e^{-i\tilde{\varepsilon}(k)|t|}dt = 0. \qquad (1.2)$$

Going over to derivatives with respect to the electron-hole operators $a_{k,s}$ (just as this was done in references 1 and 2), we rewrite (1.2) in the form

$$2\xi(k)u_kv_k = (u_k^2 - v_k^2)\sum_{k'} u_{k'}v_{k'}Q(k,\ k'), \qquad (1.3)$$

where u_k, v_k are the parameters of the canonical transformation from a to α, and

$$\xi(k) = E(k) - \lambda + i\int_{-\infty}^{\infty} e^{-i\tilde{\varepsilon}(k)|t|}\left\{\left\langle \frac{\delta^2 S}{\delta a_{k,+}(0)\,\delta a^+_{k,+}(t)} \right\rangle_c - \left\langle \frac{\delta^2 S}{\delta a^+_{-k,-}(0)\,\delta a_{-k,-}(t)} \right\rangle_c\right\} dt,$$

$$Q(k,\ k') = Q_c(k,\ k') + Q_{ph}(k,\ k'),$$

$$Q_c(k,\ k') = \begin{cases} q(k,\ k') \text{ for } k > k_F \\ q(k',\ k) \text{ for } k < k_F \end{cases}, \qquad (1.4)$$

$$q(k,\ k') = i\int_{-\infty}^{\infty} d\tau d\theta dt \exp\{-i\tilde{\varepsilon}(k)\,\tau - \theta - i\tilde{\varepsilon}(k')\,t\} \times \left\langle \frac{\delta^4 S_c}{\delta a_{k',+}(0)\,\delta a_{-k',-}(t)\,\delta a^+_{k,+}(\tau)\,\delta a^+_{-k,-}(\theta)} \right\rangle_c,$$

$$Q_{ph}(k,\ k') = \frac{2\tilde{g}^2(q)\,\tilde{\omega}(q)}{\tilde{\omega}(q) + \tilde{\varepsilon}(k') + \tilde{\varepsilon}(k)}\Lambda(k,\ k',\ q)\,\Lambda \times (-k,\ -k',\ -q),\quad q = k - k'. \qquad (1.5)$$

The details of this calculation are contained in reference 2. We abide also by the notation used in references 1 and 2.

The vertex function Λ is defined by Eq. (3.15) of reference 2. It will be essential for us in the following that in the region of the infrared Coulomb singularity for $q^2 \sim 0$ it has the form

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$$\Lambda(k, k'; q) = \frac{1}{V\overline{2V}}\left\{1 - \frac{\nu(q) F(q)}{2}\sum_s \int_{-\infty}^{\infty} dt \Psi_{s,q}(t)\right\}. \quad (1.6)$$

Here $\nu(q)$ is the kernel of the Coulomb interaction,

$$F(q) = \frac{4}{V}\sum_{\substack{l+q>k_F \\ l<k_F}} \frac{1}{\tilde{\varepsilon}(l+q) + \tilde{\varepsilon}(l)},$$

and $\Psi_{s,q}(t)$ is the coefficient for the main part of the fourth variational derivative

$$\left\langle \frac{\delta^4 S_c}{\delta a^+_{l,s}(\tau)\, \delta a^+_{k,+}(\theta)\, \delta a_{l+q,s}(0)\, \delta a_{k',+}(t)} \right\rangle_c = \frac{i\nu(q)}{V}\delta(\tau)\,\delta(t-\theta)\,\Psi_{s,q}(t). \quad (1.7)$$

2. THE PROBLEM OF THE SUMMATION OF THE COULOMB SINGULARITIES

The expressions (1.5), (1.6), and (1.7) which we have obtained for the kernels Q_c and Q_{ph} of the integral compensation equation contain fourth variational derivatives of the Coulomb matrix S_c, the expressions for which can be analyzed completely in the approximation of a high-density electron gas when the effective parameter for the Coulomb interaction $\nu(k_F)N/V$ is small compared to the electron energy at the Fermi surface $k_F^2/2m$, i.e., when

$$\frac{4\pi\hbar^2 e^2 N}{k_F^2 V} \Big/ \frac{k_F^2}{2m} = \frac{8}{\pi^3} r_s\alpha \ll 1. \quad (2.1)$$

Here $\alpha = (4/9\pi)^{1/3}$, and r_s is a dimensionless interelectron distance expressed in units of the Bohr radius.

In that case the Coulomb energy is small compared to the kinetic energy everywhere except in the region of small momentum transfers:

$$q^2 \ll k_F^2 \quad (2.2)$$

the so-called "infrared region." We can therefore use the usual perturbation theory everywhere except in the infrared region. In the infrared region, where the effective expansion parameter $r_s k_F^2/q^2$ is not small, it is necessary to sum the infinite series of "main" Coulomb terms which are proportional to the powers $(r_s k_F^2/q^2)^n$.

We meet here with a situation which is well-known in relativistic quantum electrodynamics where, notwithstanding the smallness of the dimensionless coupling constant (the fine structure constant $e^2/4\pi = 1/137$) in the so-called "ultraviolet" and "infrared" regions of the momentum variables, the actual expansion parameter is a product of $e^2/4\pi$ and a large logarithm. In quantum electrodynamics there are several well-known methods that allow us to sum the infinite sequence of main terms. We have here in mind the method of the summation of the main diagrams by Landau, Abrikosov, and Khalatnikov[3] and the method of the renormalization group.[4]

There are also a few methods, proposed in quantum statistics, which give similar results. The procedure of summing the main Coulomb diagrams developed by Gell-Mann, Brueckner. and Sawada[5] is, thus, essentially equivalent to the method stated in reference 3. The well-known method of approximate second quantization[6] in the problem of the high-density electron gas leads also to similar results.[1,7] Finally, one can use for the summation of the Coulomb singularities in the Green functions the technique of the renormalization group. (Such a possibility was first indicated in reference 8.)

An important advantage of the renormalization group method is its regularity. It will be shown that the first approximation of the renormalization group method leads to the same equations as result from the summation of the main diagrams (similar to what happens in quantum field theory) and to the equations of the method of approximate second quantization.

The results of higher approximations in the renormalization group method may be of interest for a study of the problem of extending the region of applicability of the high-density electron gas approximation.

3. THE RENORMALIZATION GROUP IN THE PROBLEM OF THE COULOMB INTERACTION BETWEEN ELECTRONS

The possiblity of using the method of the renormalization group in the problem of the Coulomb interaction between electrons is based upon (see references 8 and 9 for more detail) the group character of a finite multiplicative transformation of the basic quantities: the one-electron Green function G, the two-electron (four-vertex) Green function Γ, and the dimensionless Coulomb interaction parameter r [see (2.1)]

$$G \to G' = z_2 G, \quad \Gamma \to \Gamma' = z_1^{-1}\Gamma, \quad r \to r' = z_1 z_2^{-2} r. \quad (3.1)$$

The meaning of the transformation (3.1) consists in the fact that the totality of the quantities (G', Γ', r') describes the same physical picture as the triplet (G, Γ, r).

The main instrument of the renormalization group method are the Lee differential equations. To get these we must first of all write down the functional group equations corresponding to (3.1).

As a first step we must choose a representation of the Green functions G and Γ.

For this purpose we go over to the symmetric energy-momentum representation of the electron operators (see, for instance, reference 10)

$$a_s(p) = a_s(p_0, \mathbf{p}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ip_0\tau} a_{\mathbf{p},s}(\tau)\, d\tau,$$

$$a_s^+(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ip_0\tau} a_{\mathbf{p},s}^+(\tau)\, d\tau. \qquad (3.2)$$

The chronological pairing has the following form in that representation

$$\overbrace{a_s(p)\, a_\sigma^+(k)} = i\delta(p-k)\,\delta_{s\sigma} G_0(p),$$

$$\delta(p-k) \equiv \delta_{\rho k}\delta(p_0 - k_0),$$

$$G_0(p) = 1/[p_0 - |\tilde{\varepsilon}(\mathbf{p})| + i\alpha \operatorname{sign}\tilde{\varepsilon}(\mathbf{p})]. \qquad (3.3)$$

In the case under consideration we get for the total Green function, because of the definition of the renormalized energy $\tilde{\varepsilon}$,

$$\langle T a_s(p)\, a_\sigma^+(k)\, S\rangle_0 / S_0 = i\delta(p-k)\,\delta_{s\sigma} G(p)$$

also

$$G(p) = s(p)/[p_0 - |\tilde{\varepsilon}(\mathbf{p})| + i\alpha \operatorname{sign}\tilde{\varepsilon}]. \qquad (3.4)$$

The two-electron (four-vertex) Green function Γ is defined as

$$i\left\langle \frac{\delta^4 S}{\delta a_{s_1}(p_1)\,\delta a_{s_2}(p_2)\,\delta a_{\sigma_1}^+(k_1)\,\delta a_{\sigma_2}^+(k_2)} \right\rangle_c$$

$$= \delta(p_1 + p_2 - k_1 - k_2)\,\Gamma_{s_1 s_2 \sigma_1 \sigma_2}(p_1, p_2, k_1, k_2).$$

Here

$$\Gamma_{s_1 s_2 \sigma_1 \sigma_2}(p_1, p_2, k_1, k_2)$$

$$= \frac{i}{2\pi} \int_{-\infty}^{\infty} d\tau dt_1 dt_2 \exp\{-ip_2^0\tau + ik_1^0 t_1 + ik_2^0 t_2\}$$

$$\times \left\langle \frac{\delta^4 S}{\delta a_{s_1\mathbf{p}_1}(0)\,\delta a_{s_2\mathbf{p}_2}(\tau)\,\delta a_{\sigma_1\mathbf{k}_1}^+(t_1)\,\delta a_{\sigma_2\mathbf{k}_2}^+(t_2)} \right\rangle_c,$$

$$p_1^0 + p_2^0 = k_1^0 + k_2^0, \qquad \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{k}_1 + \mathbf{k}_2. \qquad (3.5)$$

The function Γ has a simple matrix structure

$$\Gamma_{s_1 s_2 \sigma_1 \sigma_2}(p_1, p_2, k_1, k_2) = \delta_{s_2\sigma_1}\delta_{s_1\sigma_2}\Gamma(k_1, k_2, p_1, p_2) - \delta_{s_1\sigma_1}\delta_{s_2\sigma_2}\Gamma(k_1, k_2, p_2, p_1).$$

In our case of the infrared Coulomb asymptotic behavior, the one-electron Green function G has no singularity, and we can thus put s(p) = 1. If we also take into account that for the compensation equation only the vicinity of the Fermi surface is of importance, we can restrict our consideration to the function Γ with $\mathbf{k}_1^0 = \mathbf{k}_2^0 = \mathbf{p}_1^0 = \mathbf{p}_2^0 = 0$, i.e.,

$$\Gamma(\mathbf{k}_1, \mathbf{k}_2, \mathbf{p}_1, \mathbf{p}_2)$$

$$= \frac{i}{2\pi} \int_{-\infty}^{\infty} d\tau dt_1 dt_2 \left\langle \frac{\delta^4 S_c}{\delta a_{\mathbf{p}_1,+}(0)\,\delta a_{\mathbf{p}_2,-}(\tau)\,\delta a_{\mathbf{k}_1,-}^+(t_1)\,\delta a_{\mathbf{k}_2,+}^+(t_2)} \right\rangle_c \qquad (3.6)$$

with $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{k}_1 + \mathbf{k}_2$. Indeed, we get from Eq. (1.5)

$$q(k, k') = -2\pi\Gamma(-\mathbf{k}, \mathbf{k}, \mathbf{k}-\mathbf{q}, \mathbf{q}-\mathbf{k}). \qquad (3.7)$$

On the other hand, it follows from (1.6) and (1.7) that

$$\Lambda(k, k'; q) = \{1 - 2\pi V F(\mathbf{q})\,\Gamma(\mathbf{q})\}/\sqrt{2V}, \qquad (3.8)$$

where $\Gamma(\mathbf{q})$ is the "main" part of the function $\Gamma(l+\mathbf{q}, \mathbf{k}-\mathbf{q}, \mathbf{k}, l)$, in the limit of small $\mathbf{q}$ independent of $\mathbf{k}$ and l.

In the lowest order of perturbation theory

$$\Gamma_0(k, l, l+q, k-q) = \nu(q)/2\pi V.$$

It is therefore convenient to introduce a new function g defined by the relation

$$\Gamma(k, l, l+q, k-q) = \frac{\nu(q)}{2\pi V} g(k, l, l+q, k-q). \qquad (3.9)$$

This function possesses the following important properties:

a) It transforms under the transformation (3.1) as

$$g \to g' = z_1^{-1} g; \qquad (3.10)$$

b) It tends to a constant in the limit of switching off the Coulomb interaction.

c) If it is written as a perturbation-theory series, it contains terms proportional to powers of the ratio $(r_s k_F^2/q^2)$.

In such a case it is not difficult to write down the functional group equation for the vertex function g in the infrared region. To do this we shall introduce a normalizing momentum λ and go over to dimensionless variables

$$g(q^2, k_F^2, \ldots, r) = f(q^2/\lambda^2, k_F^2/\lambda^2, r).$$

The dots indicate here the unimportant momentum variables k^2, l^2, $(k-l)^2, \ldots$. We must also include in their number the energy variables k_0, q_0, l_0. We can omit writing all these variables explicitly, considering them as fixed parameters. Taking (3.1) and (3.10) into account we get

$$f(q^2/\lambda_1^2, k_F^2/\lambda_1^2, r_1) = z_1^{-1} f(q^2/\lambda^2, k_F^2/\lambda^2, r), \quad r_1 = z_1 r. \qquad (3.11)$$

Choosing the normalizing momentum λ in such a way that

$$f(1, y, r) = 1, \qquad (3.12)$$

we go over to the functional group equation

$$f(x, y, r) = f(t, y, r) f(x/t, y/t, rf(t, y, r)). \quad (3.13)$$

Differentiating (3.12) with respect to x and afterwards putting t = x we get Lee's equation

$$\frac{\partial f(x, y, r)}{\partial x} = \frac{f(x, y, r)}{x} \Phi\left(\frac{y}{x}, rf(x, y, r)\right), \quad (3.14)$$

where the function

$$\Phi(y, r) = \frac{\partial f(x, y, r)}{\partial x}\Big|_{x=1}, \quad (3.15)$$

according to the usual correspondence requirements must be evaluated by perturbation theory.

4. THE INFRARED ASYMPTOTIC BEHAVIOR OF THE VERTEX FUNCTION

The perturbation-theory evaluation of the function f which enters into the right-hand side of (3.15) must be performed with account of the multiplicative arbitrariness of (3.11), which insures the satisfying of the normalization condition (3.12).

In this way we get, up to terms of the third order,

$$f_{\text{pert.th.}}(x, y, r) = 1 - ar\left(\frac{y}{x} - 1\right) + br^2\left(\frac{y}{x} - 1\right)^2 - cr^2\left(\frac{y}{x} - 1\right) + \cdots, \quad (4.1)$$

where a, b, and c are certain numbers (which may depend on the above-mentioned parameters). Substituting (4.1) into (3.15) and (3.14) and evaluating the quadrature, we get successively

$$\Phi(y, r) = ary + cr^2 y,$$
$$\frac{df}{f^2(1 + rfc/a)} = ary\frac{dx}{x^2},$$
$$r\frac{c}{a}\ln\left|\frac{a + rcf}{(a + rc)f}\right| + 1 - \frac{1}{f} = ary\left(1 - \frac{1}{x}\right). \quad (4.2)$$

We obtain here for f a transcendental equation which can be solved by successive approximations, taking into account the fact that r is small. This gives

$$f(x, y, r) = \left\{1 + ary\left(\frac{1}{x} - 1\right) + r\frac{c}{a}\ln\left|\frac{1 + ary(1/x - 1) + rc/a}{1 + rc/a}\right|\right\}^{-1}. \quad (4.3)$$

We note that the constant b in (4.1) does not enter into Eqs. (4.2) and (4.3). The point is that this constant is not independent. Expanding (4.3) in a power series in r and comparing it with (4.1), we find that

$$b = a^2. \quad (4.4)$$

We must now still perform the transition to the usual unnormalized function f_0, which depends on the observed value of the coupling constant $r = r_s$ and which does not contain the normalization momentum λ.

Perturbation theory gives for f_0

$$f_0^{\text{pert. th.}}(q^2, r_s) = 1 - ar_s\left(\frac{k_F^2}{q^2} + d\right) + a^2 r_s^2\left(\frac{k_F^2}{q^2} + d\right)^2 - cr_s^2\frac{k_F^2}{q^2} + \cdots. \quad (4.5)$$

Here d is a constant, just as a and c.

To perform the above-mentioned transition we use a standard method (see Sec. 42.3 of reference 4) based upon the property of the invariant charge rf:

$$rf(q^2/\lambda^2, k_F^2/\lambda^2, r) = r_s f_0(q^2, k_F^2, r_s). \quad (4.6)$$

Putting $q^2 = \lambda^2$ in (4.6) we get for r the explicit expression

$$r = r_s f_0^{\text{pert.th.}}(\lambda^2, k_F^2, r_s).$$

Substituting this value into the left-hand side of (4.6) and taking (4.3) into account we obtain, after some simple calculations,

$$f_0(q^2, r_s) = \left[1 + ar_s\left(\frac{k_F^2}{q^2} + d\right) + r_s\frac{c}{a}\ln\left(1 + ar_s\frac{k_F^2}{q^2}\right)\right]^{-1}. \quad (4.7)$$

One can easily verify that (4.5) is just the expansion of (4.7) in powers of small r_s.

5. DISCUSSION OF THE RESULTS

Let us discuss Eq. (4.7) which we have obtained and which describes the behavior of the Coulomb four-vertex function in the infrared region.

Recalling that according to (3.9)

$$\Gamma(q) = \frac{\nu(q)}{2\pi V} f_0(q^2, r_s) = \frac{2e^2}{V|q|^2} f_0(q^2, r_s), \quad (5.1)$$

we consider the expression

$$Q^{-2} = \frac{1}{q^2} f_0(q^2, r_s) = \left[q^2 + ar_s k_F^2 + q^2 ar_s d + r_s\frac{c}{a}q^2\ln\left(1 + ar_s\frac{k_F^2}{q^2}\right)\right]^{-1}. \quad (5.2)$$

In the limits of small r_s and q^2 we get

$$Q^{-2} = \frac{1}{q^2 + ar_s k_F^2} = \frac{1}{q^2 + q_s^2}, \quad (5.3)$$

where $q_s = 0.814\, r_s^{1/2} k_F$ is the inverse Thomas-Fermi screening length.

In the lowest approximation (in powers of r_s), Eq. (5.2) leads to the well-known formula of the Coulomb screening. We have thus shown that the results of the summation of the main Coulomb diagrams by Gell-Mann, Brueckner, and Sawada[5] can be obtained by the renormalization group method by far simpler means.

We note also that substituting (5.3) into (5.1) and into (3.8) and taking into account that

$$\lim_{q^2\to 0} q^2 F(q)\nu(q) = q_s^2,$$

we get*

$$\Lambda(q) = \frac{1}{\sqrt{2V}}\left\{1 - \frac{q_s^2}{q^2+q_s^2}\right\} = \frac{1}{\sqrt{2V}}\frac{q^2}{q^2+q_s^2}. \qquad (5.4)$$

Equations (5.3) and (5.4) are the same as the results of Sec. 6.2 of reference 1 obtained there by qualitative arguments.

The significance, however, of Eqs. (5.2) and (4.7) and also of the possibilities of the renormalization group method in the problem of interacting electrons is not exhausted by this. The renormalization group technique gives us a regular method to improve the asymptotic properties of the expansions of the usual perturbation theory. This fact is well-known in quantum field theory. The present investigation illustrates this fact for quantum statistics.

Equations (4.7) and (5.2) appropriately generalized to the case where the energy arguments are not equal to zero are thus the second approximation to the results of reference 5. A further generalization of these equations to higher orders can very simply be performed by means of taking higher terms in powers of r_s into account in (4.1).

Such a generalization can be of interest, for instance, to make the criterion for superconductivity (see Sec. 5.4 of reference 1) more precise and also to make the expansions for the correlation energy more precise in the region of not-very-small r_s.

We note in this connection that at the last moment we learned of the thesis by Dubois[12] in which there is an attempt to make the formulas of reference 5 more exact by summing the main diagrams of the second order. Without going into a detailed comparison, we note that the structure of the expressions obtained by Dubois[12] [Eqs. (2.7), (2.5), and (A.3) of that paper] correspond at first sight to the following substitution into (4.7);

$$r_s\frac{c}{a}\ln\left(1 + ar_s\frac{k_F^2}{q^2}\right) \to r_s^2 c\frac{k_F^2}{q^2},$$

which is valid only for small r_s and not too small q^2.

*This formula is in agreement with the result of reference 11.

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