ON THE ANALYTIC PROPERTIES OF VERTEX PARTS IN QUANTUM FIELD THEORY

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A general method is developed, on the basis of the diagram technique, for finding the singularities of quantities involved in quantum field theory.

IN recent years many papers have been devoted to the so-called dispersion relations. As is well-known, these relations are expressions of analytic properties of quantities involved in quantum field theory. Therefore the main problem is that of the location of the singularities of the quantities in question. As has recently been discovered,[1,2] the most effective method for studying the location and character of the singularities of vertex parts is the direct examination of diagrams. It is often supposed that a treatment by means of diagrams is not sufficiently convincing, since it would seem to be associated with the use of perturbation theory, unlike other methods that would seem to be more rigorous. Actually this belief is based on a misunderstanding. Since a rigorous theory using a Hamiltonian makes the interaction zero, the only completely rigorous dispersion relation in this theory is $0 = 0$. In setting ourselves the problem of studying the analytic properties of the quantities of quantum field theory, we, in fact, go beyond the framework of the existing theory. In this connection the assumption is automatically made that there exists a theory not making the interaction zero, and not using ψ-operators and Hamiltonians, but retaining the diagram technique. Therefore the use of the diagram technique in the derivation of dispersion relations is actually the only consistent method, since if we renounce the diagram technique, the very statement of the problem loses its meaning.

The diagram method is by no means equivalent to perturbation theory, since in it one treats as particles all particles that are stable from the point of view of the strong interactions, independently of whether they are "simple" or "complex." In fact, in such a treatment the first steps are made toward the construction of a new diagram technique that will be a generalization of the old one. This technique must be the basis of the future theory. Of course, the applicability of the diagram technique in such a form in the future theory is itself a hypothesis, and the testing of the results so obtained will also be a test of the hypothesis itself.

Unfortunately, the expressions obtained from the consideration of the more complicated diagrams become very lengthy, and this makes their study difficult. It can be shown, however, that such a study can be carried through in general form, and that it is much simplified by a suitable graphical representation.

An arbitrary diagram represents a certain integral

$$\int \frac{B d^4 k \, d^4 l \ldots}{A_1 A_2 A_3 \ldots}, \tag{1}$$

where

$$A_i = m_i^2 - q_i^2, \tag{2}$$

q_i is a certain four-momentum, corresponding to a given line in the diagram; m_i is the mass of the corresponding particle and B is a certain polynomial in the vectors q_i. According to the well-known method of Feynman, we can write

$$\frac{1}{A_1 A_2 \ldots A_n}$$
$$= (n-1)! \int_0^1 \ldots \int_0^1 \frac{d\alpha_1 \, d\alpha_2 \ldots d\alpha_n \cdot \delta (\alpha_1 + \alpha_2 + \ldots + \alpha_n - 1)}{(\alpha_1 A_1 + \alpha_2 A_2 + \ldots + \alpha_n A_n)^n}. \tag{3}$$

The expression $\alpha_1 A_1 + \alpha_2 A_2 + \ldots$ in the denominator is a polynomial of the second degree in the variables of integration $k, l, \ldots$. By a transformation of the variables of integration, we can always eliminate from this polynomial the terms linear in $k, l, \ldots$; when this has been done we get

$$\alpha_1 A_1 + \alpha_2 A_2 + \ldots = \varphi + K(k', l', \ldots). \tag{4}$$

Here K is a homogeneous quadratic form in the new variables of integration with coefficients depending only on the parameters α_i, and φ is an inhomogeneous quadratic form in the vectors p_i that characterize the external lines of the diagram in question.

Let us confine ourselves to the case of real

values of both the squares and also the scalar products of the vectors p_i. It is easily verified that for positive values of φ the integral over k, l, ... is a real quantity (in the case of spinor functions, a self-adjoint spinor), since the substitutions $k_4 \to i\kappa$ and $l_4 \to i\lambda$ make the quadratic form K positive definite (in virtue of the fact that all the α_i are positive). Therefore, if $\varphi > 0$ for all values of the α_i, the vertex part is real; conversely, if for some values of the α_i the form φ is negative, the vertex part becomes complex. The nearest singularity of the vertex part is obviously located at the values of the p_i for which φ vanishes for definite values of all the quantities α_i and is positive for all other values of the α_i; in other words, singularities correspond to the vanishing of the minimum value of φ, regarded as a function of the α_i. If we are to discuss the singularities in the complex region, then we must consider an arbitrary extremum of the function φ. We note that since φ is a homogeneous function of the first degree in the variables α_i, in finding the conditions for the existence of an extremum equal to zero we can omit the condition $\Sigma\alpha_i = 1$.

Let us denote the quantity $\alpha_1 A_1 + \alpha_2 A_2 + \ldots$ by the letter f. Since K is a quadratic form in the variables k', l', ... , it is clear that φ is the value of the function f under the subsidiary conditions

$$\partial f / \partial k' = \partial f / \partial l' = \ldots = 0,$$

or, since k differs from k' by a constant vector, these conditions can be written in the form

$$\partial f / \partial k = \partial f / \partial l = \ldots = 0. \tag{5}$$

In finding the conditions for a minimum we must take into account the fact that the quantities α_i are positive. From this it follows that for each quantity α_i we must have either the condition $\partial\varphi/\partial\alpha_i = 0$ or the condition $\alpha_i = 0$. In the latter case it is obvious that for the nearest singularity $\partial\varphi/\partial\alpha_i > 0$. Furthermore we have by definition

$$\frac{\partial\varphi}{\partial\alpha_i} = \frac{\partial f}{\partial\alpha_i} + \frac{\partial f}{\partial k}\frac{\partial k}{\partial\alpha_i} + \frac{\partial f}{\partial l}\frac{\partial l}{\partial\alpha_i} + \cdots$$

Since, according to Eq. (5), all the $\partial f/\partial k$ are equal to zero, it follows that the condition $\partial\varphi/\partial\alpha_i = 0$ is equivalent to the condition $\partial f/\partial\alpha_i = 0$, i.e., by the definition of f,

$$A_i = 0. \tag{6}$$

Thus the singularity of the vertex part can be obtained by solving simultaneously the conditions $A_i = 0$ (or $\alpha_i = 0$) under the subsidiary conditions

$$\sum_i \alpha_i \frac{\partial A_i}{\partial k} = \sum_i \alpha_i \frac{\partial A_i}{\partial l} = \ldots = 0. \tag{7}$$

Here it is essential that these equations must have solutions with positive α_i.

Thus regarding each line of a Feynman diagram, it can be declared that it either satisfies the condition $q_i^2 = m_i^2$, or else drops out of the treatment altogether (when $\alpha_i = 0$). In the latter case, the singularity in question can be ascribed not to the given diagram, but to the diagram in which the i-th line is absent, i.e., the vertices it connects are merged. Therefore in the analysis of the singularities of diagrams, it suffices to confine ourselves to the case in which all $\alpha_i \neq 0$.

It is easy to see that condition (7) can be written in the form $\Sigma\alpha_i q_i = 0$, where the summation is taken not over all the lines of the Feynman diagram, but over any set of lines forming a closed contour, with the directions of the vectors q_i corresponding to a direction of passage around the contour. The positive nature of the coefficients means that if one thinks of the vectors q_i as directions of forces, the possibility of satisfying Eq. (7) implies the possibility of choosing the magnitudes of these forces so that they are in equilibrium.

By means of this method one can analyze the singularities that occur with comparative ease. Let us begin with an examination of diagrams for the Green's function. In this case, all the vectors q determined from the equations that have been stated are obviously parallel to the vector p.

Let us consider the diagram of Fig. 1. Writing the formula $\Sigma\alpha_i q_i = 0$ for the contour formed by any pair of lines, and recalling that the quantities α_i are positive, we come to the conclusion that all of the vectors q_i have the same sense relative to the vertex of the diagram. Using the fact that the lengths of the vectors are equal to the corresponding masses, we get without difficulty the obvious result $p^2 = (\Sigma m_i)^2$ for the singularities. We note that in the case of Green's functions, it is superfluous to consider any other type of diagram, for example, such as that shown in Fig. 2. In fact, the number of equations for the determination of the quantities α_i is equal to the number of independent contours in the diagram, so that in the example shown it is two. At the same time, the total number of quantities α_i in this example is five; it is clear from this that one of them can be set equal to zero, and the result is that the diagram reduces to one of the diagrams of the type already considered. We note that all of these arguments are also equally applicable to diagrams of the type shown in Fig. 3, where the role of the momen-
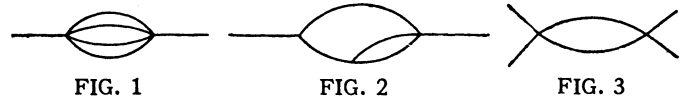

FIG. 1　　　　FIG. 2　　　　FIG. 3

tum is played by the corresponding sum of momenta.

Let us now go on to the vertex part with three external lines. Since, by the conservation laws, these three lines represent three vectors lying in one plane, and the vectors k, l ... are determined from Eqs. (6) and (7), it is clear that these latter vectors lie in the same plane. Thus in this case the problem reduces to the consideration of a plane system of vectors.

Let us consider first the simplest example, which has been analyzed in detail by Karplus, Sommerfield, and Wichmann[3] (Fig. 4). It is easy to see that the relation between the vectors is represented by the scheme shown in Fig. 5. The condition (7) obviously requires that the point 0 lie inside the triangle. The tacit assumption here is that all the vectors have the properties of Euclidean (not pseudo-Euclidean) vectors; this is easily shown for the nearest singularities.
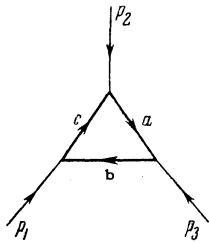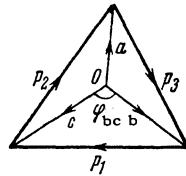

FIG. 4   FIG. 5

To begin with, we show how to formulate analytically the relations that are expressed graphically by diagrammatic schemes. First, let us consider the scheme of Fig. 5, which corresponds to the diagram of Fig. 4. If we introduce the unit vectors $n_a = q_a/m_a$, $n_b = q_b/m_b$, $n_c = q_c/m_c$, the condition (7) can be written in the form

$$\beta_a n_a + \beta_b n_b + \beta_c n_c = 0,$$

where $\beta_i = \alpha_i m_i$ are obviously also positive quantities. Projecting these equations successively on the vectors n_a, n_b, n_c and introducing the notations $(n_a n_b) = \mu_c = \cos \varphi_{ab}$, etc., we get the three equations

$$\beta_a + \beta_b \mu_c + \beta_c \mu_b = 0, \qquad \beta_a \mu_c + \beta_b + \beta_c \mu_a = 0,$$
$$\beta_a \mu_b + \beta_b \mu_a + \beta_c = 0. \tag{9}$$

Equating the determinant of this system to zero gives the equation

$$1 + 2\mu_a \mu_b \mu_c = \mu_a^2 + \mu_b^2 + \mu_c^2, \tag{10}$$

which determines the position of the singularity, under the simultaneous condition that all the β's are positive, if we use the fact that μ_a, μ_b, μ_c are connected with p_1^2, p_2^2, p_3^2 by the formulas

$$\mu_a = \frac{m_b^2 + m_c^2 - p_1^2}{2m_b m_c}, \qquad \mu_b = \frac{m_a^2 + m_c^2 - p_2^2}{2m_a m_c},$$

$$\mu_c = \frac{m_a^2 + m_b^2 - p_3^2}{2m_a m_b}. \tag{11}$$

Returning to the question of the properties of the vectors for the nearest singularity, let us consider any angle, say φ_{bc}. According to Eq. (11), μ_a is a real quantity. It is larger than -1 since otherwise we would have $p^2 > (m_b + m_c)^2$; that is, we would have gone beyond the singularity defined by the dia-

FIG. 6
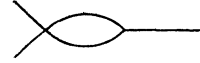
gram of Fig. 6, which is obtained from the diagram under consideration by the liquidation of one of the lines. At the same time we see from Eq. (9) that at least two of the three cosines must be negative. But a negative cosine larger than -1 always corresponds to a real angle. Therefore two of the three angles are certainly real, and since their sum is 2π, the third is also real. Thus Fig. 5, which was mentioned in reference 2, completely solves the problem.
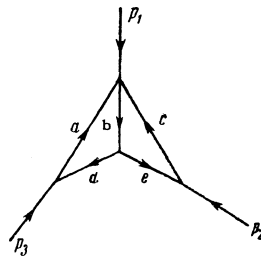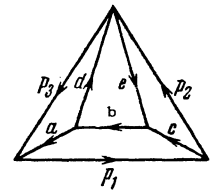

FIG. 7   FIG. 8

Let us now go on to more complicated diagrams. Let us consider, for example, the diagram shown in Fig. 7. The corresponding scheme is shown in Fig. 8. The construction of such a scheme is very simple, if we use the fact that to each point of the Feynman diagram there corresponds a polygon in the scheme, with its number of sides equal to the number of lines that come together at the point, and to each polygon of the Feynman diagram there corresponds a point in the drawing of the scheme. The condition (7) requires the corresponding location of the vectors a, b, and d on one side, and b, c, and e on the other. It is not hard to write the analytical expression of this scheme. The triangle bde, as all the parts of the drawing, is real, since the diagram involves only stable particles, and therefore the mass of each of the particles in b, d, e is less than the sum of the other two.

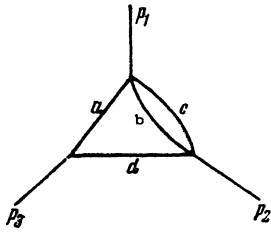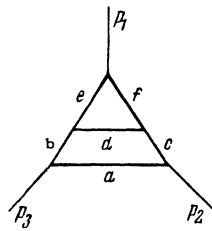In cases in which one or more of the coefficients

FIG. 9   FIG. 10

$\alpha_i = 0$, the diagram is simplified. For example, if $\alpha_c = 0$, the diagram of Fig. 7 becomes the diagram of Fig. 9, analogous to Fig. 4.

Going on to more complicated Feynman diagrams, one often finds that the case with all $\alpha_i \neq 0$ is impossible. Let us examine, for example, the diagram shown in Fig. 10. In this case the number of vector equations for α_i is two; that is, in view of the plane character of the vectors we have four equations, but the number of quantities α_i is six, and therefore one of them can be set equal to zero. The same argument applies to diagrams of the type shown in Fig. 11.
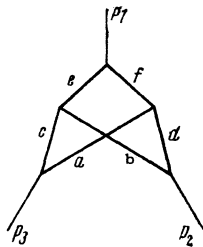

FIG. 11

It is easy to see how the results found here are changed when we concern ourselves with things beyond the nearest singularity, i. e., with the appearance of complex quantities and of singularities in quantities that are already complex. If a singularity is not the nearest one, then, generally speaking, we cannot assume that all of the angles are real, as we have done earlier for the nearest singularities.

Let us first consider the scheme of Fig. 5, corresponding to the diagram of Fig. 4. In the "Euclidean" case, with all the angles real, all the μ_i lie between -1 and $+1$. In the general case, a study of Eq. (10) shows that the corresponding surface in the space of μ_a, μ_b, μ_c consists of four parts, that intersect each other in three points: $\mu_a = 1$, $\mu_b = -1$, $\mu_c = -1$, and the points obtained by interchange of the indices a, b, c. One of these surfaces is a curved triangle bounded by the straight lines joining the three points. This triangle corresponds to the "Euclidean" case. The three other surfaces go to infinity, each beginning at the corresponding point of intersection. On each of these surfaces one of the μ's is positive and larger than unity, and the two others are negative and in absolute value also larger than unity. As has already been pointed out, these

surfaces always correspond to singular points that are not the nearest.

It is easy to see that in the diagrams under consideration, the non-Euclidean case is possible only in the diagram of Fig. 4. For example, in the scheme of Fig. 8, corresponding to the diagram of Fig. 7, the role of the vectors a, b, c is played by the vectors a, b, d on the one side, and by b, c, e on the other. As has already been shown, however, the angles in the triangle made up of the vectors b, d, e are necessarily real. Otherwise one of the particles b, d, e would be unstable. At the same time, as has just been shown, in the non-Euclidean case all the cosines must have absolute values larger than unity, and consequently all the angles must be complex.

Let us now discuss Feynman diagrams with four external lines. In this case, we obviously have to consider schemes that lie not in a plane but in space, and naturally this complicates the problem. The most important such diagrams are those with physical external lines, i. e., those in which the squares of the momenta in question are equal to the squares of the masses of the particles. The simplest singularities are associated with diagrams of the type of Fig. 12. Here the middle line can correspond either to one particle or to several particles. The singularities of such a diagram obviously correspond to

$$(p_1 + p_2)^2 = m_a^2,$$

where m_a is the sum of the masses of these particles. The one-particle case gives an isolated pole. The two-particle case is the boundary of the complex region.
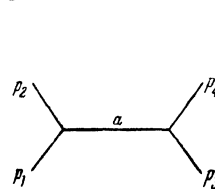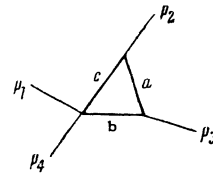

FIG. 12   FIG. 13

Figure 13 corresponds entirely to Fig. 4 in its properties, and therefore requires no further analysis.

To Fig. 14 there corresponds the scheme of Fig. 15, where four of the six edges of the tetrahedron are determined by the lengths of the corresponding
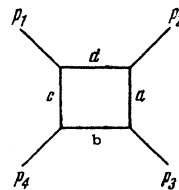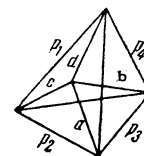

FIG. 14   FIG. 15

external lines, and the other two are equal to $p_1 + p_2$ and $p_2 + p_3$, respectively. (We note that this sort of graphical construction for this diagram occurs in the work of Karplus et al.[3]) Here, too, a detailed analysis shows that for the case of the nearest singularity, all of the angles are real, and the condition that the coefficients α are positive corresponds to the condition that the central point lie inside the tetrahedron. (The situation corresponding to the singularities in the complex region is more complicated, and we shall not discuss it.) As in the analysis of the diagrams with three external lines, it is convenient to introduce four unit vectors in the directions of the vectors a, b, c, d and find an equation connecting the cosines of the angles between them. It turns out that instead of Eq. (10) we get an equation of the fourth degree. We note the following curious fact. If we consider diagrams with four physical external lines, then as can be seen from the scheme of Fig. 5, for the diagram of Fig. 13 the following condition must be satisfied: the sum of the angle between the sides a and b in the triangle abp_3 and that between the sides a and c in the triangle acp_2, both of which are fixed by the masses of corresponding particles, must be larger than π. The analogous condition for the diagram of Fig. 14 is obviously that the sum of four angles in the triangles corresponding to the four vertices must exceed 2π. From this it follows directly that a necessary, though of course not sufficient, condition for the existence of "nontrivial" nearest singularities is the presence of an obtuse angle in at least one of the triangles, and for this it is in turn necessary that for the virtual decay of at least one of the particles an inequality of the type

$$m_2^2 > m_a^2 + m_c^2.$$

must hold. This relation obviously cannot be satisfied for either mesons or nucleons. Therefore, the nearest singularities for the scattering of these particles by each other correspond to the diagram of Fig. 12. We emphasize, however, that this does not apply to singularities in the complex region.

Let us now examine the nature of the singularities that are obtained. For this purpose we return to the basic formula (1), writing it in the form

$$\int (\varphi + K)^{-n} B d^4k' d^4l' \ldots d\alpha_1 d\alpha_2 \ldots \delta(\alpha_1 + \ldots + \alpha_n - 1).$$

Expanding φ in powers of $\alpha_i' = \alpha_i - \alpha_{0i}$, where the α_{0i} correspond to the minimum value of φ, we can write this integral in the form

$$\int (\varphi_0 + Q)^{-n} B d^4k' d^4l' \ldots d\alpha_1' d\alpha_2' \ldots \delta(\alpha_1' + \ldots + \alpha_n'), \quad (12)$$

where φ_0 is the minimum value of φ, which is equal to zero at the singularity itself (for prescribed values of the vectors for the external lines), and Q is a quadratic function both in the variables k', l', ... and in the variables α_i'. To find the character of the singularity, it suffices to confine ourselves to the values of B at $k' = l' = \ldots = 0$. If the degree of the numerator with respect to the variables of integration is lower than the degree of the expression Q^n, the integral (12) converges for large values of these variables; in other words, its value is determined by values of the variables corresponding to $Q \sim \varphi_0$; i.e., the integration occurs over small values of the variables, for which the expression (12) has sufficient accuracy. It is obvious that in this case the expression (12) can be written in the form

$$\text{const} \cdot \varphi_0^{m/2 - n}, \quad (13)$$

where m is the number of integrations. If $m \geq 2n$, the integral (12) diverges at high values, and these arguments do not apply. In order to determine the character of the singularity in this case, it is simplest to differentiate the expression (12) with respect to φ_0 as many times as necessary to make the degree of the denominator larger than that of the numerator. After this, we can use the resulting formula, which now has to be integrated the same number of times. The constants that appear in the integrations obviously give integral powers of φ_0, which have no singularity at $\varphi_0 = 0$. Accordingly we again get the result (13), except in cases in which $m/2 - n$ is zero or a positive integer. In this case instead of (13) one obviously gets

$$\text{const} \cdot \varphi_0^{m/2 - n} \ln \varphi_0.$$

We note that although we speak here of a minimum, these results are equally applicable to any extremum of φ.

The quantity n is the number of internal lines in the Feynman diagram; the number of vectors k over which one integrates is equal to the number ν of independent contours that make up the diagram in question. Accordingly,

$$m = 4\nu + n - 1.$$

It follows from this that the character of the singularity is given by the expression

$$\varphi_0^{2\nu - (n+1)/2},$$

or if $2\nu - (n + 1)/2$ is zero or a positive integer, by the expression

$$\varphi_0^{2\nu - (n+1)/2} \ln \varphi_0.$$

The quantity φ_0 is obviously proportional to the distance of the point in question from the hypersurface in the space of p_i^2, $(p_i p_k)$, ... on which the singu-

lar points are located. In counting the number of contours we must include also the "diangles" that occur when several particles go along a single line. For example, in the diagram of Fig. 9, $\nu = 2$, and, since n = 4, the singularity has the character $\varphi_0^{3/2}$.

We note that instead of the number of independent contours, one can use the more convenient number of vertices. In fact, the number of independent contours, i. e., the number of independent integrations, is equal to the number of lines minus the number of subsidiary conditions. This last number is equal to one less than the number of vertices, since the δ-function goes over into the final result. Thus we have

$$\nu = n - v + 1$$

(where v is the number of vertices),* and the degree of the singularity can be written in the form $(3/2)(n+1) - 2v$.

Let us formulate briefly the general rules for finding the singularities. One considers various diagrams with the given external lines. In the construction of such diagrams an arbitrary number of lines can meet at a vertex, subject of course to the conservation laws (for example, an odd number of π-meson lines cannot meet at a point). All particles stable under the strong interactions can occur as lines. After drawing a diagram one studies the scheme constructed according to the principle of replacing polygons in the diagram by vertices in the scheme, from each of which there emerges a corresponding number of straight line-segments. The lengths of all the internal lines in the scheme are equal to the corresponding masses. At the essential intersections in the scheme (those that arise from polygons in the diagram), the conditions $\Sigma \alpha_i q_i = 0$ must hold, where the q_i are the vectors emerging from a given intersection, and all of the α_i are positive. Nearest singularities correspond to schemes in Euclidean space.

In the application of these results to the scattering amplitude, a number of circumstances must be kept in mind. We shall consider the scattering amplitude as a function of one variable x (this can be, for example, the total energy or the momentum transfer), regarding all other variables as given. As is well-known, the integrals considered here define functions whose values in the upper and lower half-planes are connected by the relation $f(x^*) = f^*(x)$; in other words, above and below the axis we are dealing with essentially different functions, which are by no means analytic continuations of each other, and at real values of x we have, in general, a discontinuity. The scattering amplitude, which is obtained by using the Feynman rules for passage around poles, in general has the form $a(x + i\delta) + b(x - i\delta)$ (δ is infinitely small). The analytic properties considered earlier refer to the function $a(x + i\delta)$, continued into the upper half-plane of the variable x, and to the function $b(x - i\delta)$, continued into the lower half-plane. The behavior of the analytic continuation of the function $a(x + i\delta)$ into the lower half-plane, and that of the continuation of the function $b(x - i\delta)$ into the upper half-plane, are by no means evident from the foregoing discussion.

In the "foreign" half-plane the function a or b can have any singularities, located in any way, and they, in general, cannot be determined from any general considerations. For example, as is well-known, in addition to the singularity associated with the formation of a deuteron, the proton-neutron scattering amplitude has a singularity associated with the so-called virtual state of this system, which does not correspond to any real particle, and which indeed lies in the "foreign" half-plane of the total energy of the system. Another example is the well-known resonance in the scattering of π-mesons by nucleons, which also corresponds to a singularity in the "foreign" half-plane, and obviously at a complex value of the total energy of the system. It is clear that such singularities in principle cannot be predicted from general considerations, but can be obtained only from a theory that gives concrete expressions for the scattering amplitude.

The problem is greatly simplified in cases in which there is a region of values of x for which the amplitude in question is real. Then in the complex plane we have two segments of the real axis, and it is easy to see that the function $a(x + i\delta)$ has only the right-hand segment, and the function $b(x - i\delta)$ only the left-hand segment. If instead of the scattering amplitude we consider the quantity $a + b^*$, for which we have only to change the sign of the imaginary part of the amplitude on the left-hand segment, we get a function that has no singularities in the upper half-plane, and which leads to the usual dispersion relations.

In conclusion I would like to thank L. B. Okun', A. P. Rudik, and Ya. A. Smorodinskiĭ for many helpful comments.

*My attention was called to this fact by L. Okun' and A. Rudik.

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