RADIATIVE DEVIATIONS FROM THE COULOMB LAW AT SMALL DISTANCES

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The radiative corrections to the Dirac equation in a Coulomb field are examined for distances $r \ll \hbar/mc$. The calculations are carried to the first order in $e^2/\hbar c$ and the second order in $Ze^2/\hbar c$. The resulting change in the Coulomb singularity of the wave functions is small and is hard to distinguish from the effects of the finite size of the nucleus.

1. For an electron moving in an external field the radiative corrections are made up of two qualitatively different effects: the polarization of the electron-positron vacuum by the external field, and the interaction with the fluctuations of the photon vacuum. The first of these effects strengthens the interaction, since the electron penetrates inside the screening cloud, and in the domain of applicability of perturbation theory the vacuum-polarization potential has the form:[1]

$$V_{\mathrm{pol}} = -\frac{Z\alpha^2}{r}\frac{2}{3\pi}\left(\ln\frac{1}{mr} - \frac{5}{6} - \ln\gamma\right),$$
$$\hbar = c = 1, \quad \gamma = 1.781,$$
$$r \ll 1/m, \quad \alpha = e^2 = 1/137. \tag{1}$$

The photon fluctuations, on the other hand, lead to a "trembling" of the electron, weaken the coupling of the electron with the external field, and decrease the interaction. In the region of nonrelativistic motion of the electron, $r > 1/m$, the effect of the trembling on the behavior of the electron can be described[2] by replacing the potential energy by its average value over the fluctuation motion in the photon vacuum

$$V(\mathbf{r}) \to \langle V(\mathbf{r} + \widehat{\Delta\mathbf{r}})\rangle_{\mathrm{fl}}$$
$$= -\frac{Z\alpha}{2\pi^2}\int\frac{d\mathbf{q}}{q^2}\exp\{i\mathbf{q}\mathbf{r} - q^2\langle\Delta r^2\rangle/6\}, \tag{2}$$

where

$$\widehat{\Delta\mathbf{r}} = \frac{ie}{m}\sum_{\mathbf{k}}\sqrt{2\pi/k^3}\,\mathbf{e}_{\mathbf{k}}(a_{\mathbf{k}} - a_{\mathbf{k}}^{+}),$$

$$\langle\Delta r^2\rangle = \frac{2\alpha}{\pi m^2}\int_{k_{min}}^{k_{max}}\frac{dk}{k} = \frac{2\alpha}{\pi m^2}\ln\frac{m}{\varepsilon_0}.$$

In this way, as is well known, one gets the correct result for the Lamb shift. But if one applies Eq. (2) for $r < 1/m$, then beginning at a distance $r \sim r_c = (\langle\Delta r^2\rangle)^{1/2} \sim \alpha^{1/2}/m$ the Coulomb rise of the potential is arrested, and the $1/r$ law is replaced by a constant, $1/r \to 1/r_c$. This would mean that at distances 10 times nuclear dimensions the effective potential acting on the electron has nothing in common with the Coulomb potential and has no singularity. Such a result comes naturally from the argument in question, since the potential is averaged over the region of the "trembling" of the electron, of the order of r_c, and an electron near the nucleus is constantly carried out of the region of small r by the oscillations in the photon vacuum, so that the "average over the oscillations" $\langle V(\mathbf{r} + \widehat{\Delta\mathbf{r}})\rangle$ is finite even for $r = 0$. This approach is that of the "adiabatic" problem, in which the potential for a slow motion is obtained by averaging over a fast motion. Actually the region $r < 1/m$ corresponds to ultrarelativistic motion of the electron, so that the "frequency" of its motion, $\sim 1/r$, exceeds the characteristic frequency of the virtual quanta, $\omega \sim m$, and the oscillations have little effect on the motion. We can say that in this region the strong Coulomb field damps the fluctuation motion of the electron, and reduces the "smearing out." It turns out that the effective fluctuation radius for $r < 1/m$ falls off linearly with the distance: $(\langle\Delta r^2\rangle)^{1/2} \sim r\alpha^{1/2}$. The resultant "correction to the potential" is of the order

$$\langle V(\mathbf{r} + \widehat{\Delta\mathbf{r}})\rangle - V(r) \sim \frac{\partial^2 V}{\partial r^2}\langle\Delta r^2\rangle \sim \alpha r^2\frac{V}{r^2} = \alpha V,$$

i.e., unlike the effect of vacuum polarization the correction from the photon fluctuations is small compared with V for all r. Therefore the weakening of the interaction because of the oscillations remains finite at extremely small distances, and is described by perturbation theory for arbitrary r (the asymptotic values of $\Gamma(p_1, p_2)$ and $G(p)$ for $p \to \infty$ are equal to their "null" values to accuracy $e_0^2 < 1$).[3]

It is not only as a matter of principle that the study of the radiative deviations from the Coulomb law at small r is of interest. In electron-nuclear phenomena (β decay, K capture, the conversion of high-energy quanta) the behavior of the electron wave function for $r \sim r_{nuc} \sim 10^{-13}$ cm is very important. These functions have a Coulomb relativistic singularity at $r = 0$

$$\psi \sim (r/a_0)^{\sqrt{1-Z^2\alpha^2}-1} = \exp\left\{(1-\sqrt{1-Z^2\alpha^2})\ln\frac{a_0}{r}\right\},$$

$$a_0 = 1/mZ\alpha.$$

The quantity $\ln(a_0/r_{nuc})$ is about 5 for intermediate and large Z. Noting that measurable quantities involve squares and fourth powers of the wave functions, we may suppose that a change of the singularity by a quantity of the order α will be observable and comparable with, for example, the effect of the finite size of the nucleus.

In the language of diagrams we want to find the corrected "end" of the electron line in the Coulomb field for the case of processes taking place in the region $r \gg 1/m$, that is, $p \gg m$. The result in the form of a "corrected wave function" can be substituted into the various transition amplitudes. In many-step transitions, for example conversions, additional virtual processes are possible, and then the corresponding radiative corrections must be added to those we have found here.

Calculations of the vacuum-polarization potential have been made previously (see the papers of Schwinger[1] and of Wichmann and Kroll[4]).* Therefore it suffices to find the correction associated with the "trembling." The sign of the effect will of course be opposite to that of the effect of the vacuum; there is after all still a "weakening" of the external field.

2. We start with Schwinger's equation for the motion of an electron in an external field:[5,6]

$$[i(\hat{p} - e\hat{A} - e\hat{A}_p) + m]\psi(x) + \int M(x, y)\psi(y)\,d^4y = 0. \quad (3)$$

Here A_p is the potential of the vacuum polarization, and $M(x, y)$ is the mass operator. In the first radiative approximation

$$M(x, y) = \delta m_0 \delta(x-y) - 4\pi i\alpha\gamma_\mu G(x, y)\gamma_\mu D(x-y). \quad (4)$$

Here $G(x, y)$ is the Green's function of the electron in the external field:

$$[i(\hat{p} - e\hat{A}) + m]\,G(x, y) = \delta(x-y);$$

$D(x-y)$ is the Green's function of the quanta:

$$D(x-y) = (2\pi)^{-4}\int D(k)\,e^{ik(x-y)}\,d^4k = (2\pi)^{-4}\int \frac{d^4k}{k^2}\,e^{ik(x-y)};$$

and δm_0 is the renormalization constant—the difference of the "bare" and observed masses. In the renormalization we follow Karplus and Klein[6] and Feynman.[7] Let us consider the mass operator in the first radiative approximation; then in the mass term we take the unperturbed Coulomb wave function, which satisfies the equation

$$[i(\hat{p} - e\hat{A}) + m]\,\psi_c(x) = 0 \quad (5a)$$

or in the momentum representation

$$\int d^4p_2\,[\delta(p_1 - p_2)(i\hat{p}_2 + m) - ie\hat{A}_{p_1-p_2}]\,\psi_c(p_2) = 0. \quad (5b)$$

Further let us introduce instead of the function $D(k)$ the cut-off function $D_\Lambda(k) = k^{-2}\Lambda^2/(\Lambda^2 + k^2)$, presuming that at the end of the calculations $\Lambda \to \infty$. Then the constant $\delta m_0(\Lambda)$ is chosen so that in the absence of the external field Eq. (3) would go over into the Dirac equation with the observed mass m:[7]

$$\delta m_0(\Lambda) = m_0 - m = -m\frac{3\alpha}{2\pi}\left(\ln\frac{\Lambda}{m} + \frac{1}{4}\right). \quad (6)$$

Then on using Eqs. (5) and (6) in the mass term and letting Λ go to ∞, we get an unambiguous final result, which describes the interaction of the particle with the vacuum fluctuations. The mass operator acting on the Dirac wave functions absorbs within itself in gauge-invariant form all the divergences of the proper energy and vertex parts, and gives the physical effects directly.

The interpretation of the mass term in Eq. (3) is obvious from a comparison of Eq. (3) with Eq. (2): it is the exact relativistic analogue of the expression $(\langle V(\mathbf{r} + \hat{\Delta}\mathbf{r})\rangle - V(\mathbf{r}))\Psi(\mathbf{r})$, i.e., the smearing out of the external field by the "trembling," which is valid in the relativistic domain.

Going over to the momentum representation, we get

$$M(p_1, p_2) = (2\pi)^{-4}\delta m_0(\Lambda)\,\delta(p_1 - p_2)$$

$$- 4\pi i\alpha\int d^4k\cdot\gamma_\mu G(p_1 - k, p_2 - k)\,\gamma_\mu D_\Lambda(k).$$

In what follows we shall be interested in distances $r \ll 1/m$, i.e., $p_1, p_2 \gg m$. Therefore we shall set $m = 0$ everywhere in the mass operator. At the same time the constant δm_0 also vanishes: its divergent part would in any case cancel out, and the finite part is proportional to m. The mass operator takes the form

$$M(p_1, p_2) = -\frac{\alpha}{2^6\pi^7}\int\frac{d^4k}{k^2}\frac{\Lambda^2}{\Lambda^2 + k^2}\gamma_\mu\left(\frac{1}{\hat{p} - e\hat{A}}\right)_{p_1-k,\,p_2-k}\gamma_\mu \quad (7a)$$

or

$$M(p_1, p_2) = -\frac{\alpha}{2^6\pi^7}\int\frac{d^4k}{k^2}\frac{\Lambda^2}{\Lambda^2 + k^2}\gamma_\mu\left[\delta(p_1 - p_2)\frac{1}{\hat{p}_1 - \hat{k}}\right.$$

$$\left. + \frac{1}{\hat{p}_1 - \hat{k}}\,e\hat{A}_{p_1-p_2}\frac{1}{\hat{p}_2 - \hat{k}} + \ldots\right]. \quad (7b)$$

*The effect of the finiteness of the nuclear radius, which diminishes the interaction for $r < r_{nuc}$, on the motion is opposite to that of the "incomplete screening" associated with the polarization of the vacuum. Therefore the result of reference 4, that these effects have different signs, is obvious without calculation.

For the Coulomb-field case under consideration

$$ie\hat{A}_q = Z\alpha\delta(q_0)\beta/2\pi^2 q^2.$$

After integration over the time Eq. (3) goes over into the equation

$$\left(\alpha\mathbf{p} - \frac{Z\alpha}{r} - V_{\text{pol}} + \beta m\right)\psi(\mathbf{r}) + \int \mathfrak{M}(\mathbf{r}, \mathbf{r}')\psi(\mathbf{r}')\,d\mathbf{r}' = 0, \quad (8)$$

where $\mathfrak{M}(\mathbf{r}, \mathbf{r}') = \int d\mathbf{p}\,d\mathbf{p}'\,\mathfrak{M}(\mathbf{p}, \mathbf{p}')e^{i\mathbf{p}\mathbf{r} - i\mathbf{p}'\mathbf{r}'}$, with

$$\mathfrak{M}(\mathbf{p}_1, \mathbf{p}_2) = -\frac{\alpha}{2^5\pi^6}\int \frac{d^4k}{k^2}\frac{\Lambda^2}{\Lambda^2 + k^2}\beta\gamma_\mu\left(\frac{1}{\hat{p} - \hat{k} - e\hat{A}}\right)_{\mathbf{p}_1, \mathbf{p}_2}\gamma_\mu$$

$$= -\frac{\alpha}{2^5\pi^6}\int \frac{d^4k}{k^2}\frac{\Lambda^2}{\Lambda^2 + k^2}\beta\gamma_\mu\left[\delta(\mathbf{p}_1 - \mathbf{p}_2)\frac{1}{\hat{p}_1 - \hat{k}}\right.$$

$$+ \frac{1}{\hat{p}_1 - \hat{k}} e\hat{A}_{\mathbf{p}_1 - \mathbf{p}_2}\frac{1}{\hat{p}_2 - \hat{k}}$$

$$\left. + \frac{1}{\hat{p}_1 - \hat{k}}\int d\mathbf{q}\, e\hat{A}_{\mathbf{p}_1 - \mathbf{q}}\frac{1}{\hat{q} - \hat{k}} eA_{\mathbf{q} - \mathbf{p}_2}\frac{1}{\hat{p}_2 - \hat{k}} + \ldots\right]$$

$$= \mathfrak{M}^{(0)} + \mathfrak{M}^{(1)} + \mathfrak{M}^{(2)} + \ldots;$$

$$ie\hat{A}_{\mathbf{q}} = \beta Z\alpha/2\pi^2 q^2; \qquad \hat{p}_{1,2} = \gamma\mathbf{p}_{1,2} - i\beta\varepsilon_0 \approx \gamma\mathbf{p}_{1,2}; \quad (9)$$

γ, β are the Dirac matrices; ϵ_0 is the energy of the bound state.

3. By introducing Feynman ordering indices[8] it is easy to carry out the integration over the momenta k of the quanta in Eq. (7a), and even to perform the renormalization. [Here one uses $(\hat{p} - e\hat{A})\Psi_C = 0$, so that terms of the form const $\times$ $(\hat{p} - e\hat{A})$ are dropped.]

It is, however, more difficult to "disentangle" the resulting symbolic expression, even by expansion in powers of $Z\alpha$, that to calculate the terms of the series (7b) directly. Therefore we use the expansion (9). The calculations will be carried to terms in $Z^2\alpha^2$, inclusive; as will be shown, the change of the wave function at the origin begins only with terms $\sim Z^2\alpha^2$, so that it is necessary to go to the second approximation. Since because of the presence of a numerical factor $1/2\pi$ as a common multiplier the effect is a small one, calculation of the terms $\sim Z^3\alpha^3$ and of higher orders can hardly be of interest, although experimental manifestation of the effect could be expected only for large Z.

Let us find $\mathfrak{M}^{(0)}$; to make the method (cf., e.g., reference 9) clear we shall do the calculation in detail, although the result is known.[8]

$$\mathfrak{M}^{(0)}(\mathbf{p}_1, \mathbf{p}_2) = \alpha(2\pi)^{-4}\delta(\mathbf{p}_1 - \mathbf{p}_2)\beta J_0(p_1),$$

$$J_0(p) = -i\gamma_n\frac{\Lambda^2}{\pi^2}\int_0^\infty dx\int_0^\infty dy\int_0^\infty dz\int d^4k \cdot \exp\{-i[k^2(x + y + z)$$

$$- 2kpx + \Lambda^2 y]\}(p_n - k_n)$$

$$= -\hat{p}\Lambda^2\int_0^\infty \frac{dx\,dy\,dz\,(y + z)}{(x + y + z)^3}\exp\left\{-i\left(p^2\frac{x(y + z)}{x + y + z} + \Lambda^2 y\right)\right\}.$$

Let us introduce the variable $u = y + z$, $du = dy$, and then integrate with respect to z:

$$J_0(p) = i\hat{p}\int_0^\infty dx\,du\,\frac{u}{(x + u)^3}e^{-ip^2xu/(x+u)}(1 - e^{-i\Lambda^2 u}).$$

Setting now $x = ut$, $dx = u\,dt$ and integrating with respect to u, we find

$$J_0(p) = i\hat{p}\int_0^\infty \frac{dt}{(t + 1)^3}\ln\frac{\Lambda^2 + p^2t/(t+1)}{p^2t/(t+1)} \underset{\Lambda\to\infty}{\longrightarrow} i\hat{p}\left(\ln\frac{\Lambda}{p} + \frac{3}{4}\right).$$

Thus we have

$$\mathfrak{M}^{(0)}(\mathbf{p}_1, \mathbf{p}_2) = \frac{\alpha}{(2\pi)^4}\delta(\mathbf{p}_1 - \mathbf{p}_2)\left(\ln\frac{\Lambda}{p_1} + \frac{3}{4}\right)\beta i\hat{p}_2. \quad (10)$$

According to Eq. (5b), in our case the operator $f(\mathbf{p}_1)\,\delta(\mathbf{p}_1 - \mathbf{p}_2)\,i\hat{p}_2$ is equivalent to $f(\mathbf{p}_1)\,ie\hat{A}_{\mathbf{p}_1 - \mathbf{p}_2}$, and therefore $\mathfrak{M}^{(0)}$ can also be represented in the form

$$\mathfrak{M}^{(0)}(\mathbf{p}_1, \mathbf{p}_2) = \frac{\alpha}{(2\pi)^4}\beta\, ie\hat{A}_{\mathbf{p}_1 - \mathbf{p}_2}\left(\ln\frac{\Lambda}{p_1} + \frac{3}{4}\right). \quad (11)$$

Applying the method described above, we find

$$\mathfrak{M}^{(1)}(\mathbf{p}_1, \mathbf{p}_2) = -\frac{\alpha}{2^5\pi^6}\beta\int\frac{d^4k}{k^2}\frac{\Lambda^2}{\Lambda^2 + k^2}$$

$$\times\gamma_\mu\frac{\hat{p}_1 - \hat{k}}{(p_1 - k)^2}e\hat{A}_{\mathbf{p}_1 - \mathbf{p}_2}\frac{\hat{p}_2 - \hat{k}}{(p_2 - k)^2}\gamma_\mu$$

$$= \frac{\alpha}{(2\pi)^4}i\beta e\hat{A}_{\mathbf{p}_1 - \mathbf{p}_2}\left[-\ln\frac{\Lambda}{|\mathbf{p}_1 - \mathbf{p}_2|} - \frac{1}{4} + J_1(\mathbf{p}_1, \mathbf{p}_2)\right], \quad (12)$$

where

$$J_1(\mathbf{p}_1, \mathbf{p}_2) = \int_0^\infty \frac{d\xi\,dt}{(\xi + t + 1)^2}\frac{\tfrac{1}{2}(p_1^2 + p_2^2 t) - \hat{p}_2\hat{p}_1\xi}{(p_1^2 + p_2^2 t)\xi + (\mathbf{p}_1 - \mathbf{p}_2)^2 t}. \quad (13)$$

Combining Eqs. (11) and (12), we now get the following expression for the mass operator in first approximation in α and $Z\alpha$ for $p_1, p_2 \gg m$, $|\mathbf{p}_1|$, $|\mathbf{p}_2| \gg p_0 = \epsilon_0$:

$$\mathfrak{M}_1(\mathbf{p}_1, \mathbf{p}_2) \equiv \mathfrak{M}^{(0)} + \mathfrak{M}^{(1)}$$

$$= eV_{\mathbf{p}_1 - \mathbf{p}_2}\frac{\alpha}{(2\pi)^4}\left(\ln\frac{|\mathbf{p}_1 - \mathbf{p}_2|}{p_1} + \frac{1}{2} + J_1(\mathbf{p}_1, \mathbf{p}_2)\right). \quad (14)$$

For the Coulomb field $eV_{\mathbf{p}_1 - \mathbf{p}_2} = Z\alpha/2\pi^2|\mathbf{p}_1 - \mathbf{p}_2|^2$. The gauge invariance of Eq. (14) can be checked easily: for $V_{\mathbf{p}_1 - \mathbf{p}_2} \to \delta(\mathbf{p}_1 - \mathbf{p}_2)$ the expression (14) goes to zero.

We shall not give here the cumbersome form of the operator $\mathfrak{M}_1(\mathbf{r}_1, \mathbf{r}_2)$ in the coordinate representation. We remark only that it can be seen from Eq. (9) that for the Coulomb field $\mathfrak{M}(\mathbf{r}_1, \mathbf{r}_2)$ is a homogeneous function of $\mathbf{r}_1$, $\mathbf{r}_2$ of degree -4. From Eqs. (14) and (13) it follows further that for $r_1 \to 0$, $\mathfrak{M}_1$ diverges not more strongly than as $1/r_1$. Therefore it is clear that for the values of r_1 and r_2 in question the mass operator contains $\delta(\mathbf{r}_1 - \mathbf{r}_2)/r_1$ and analogous expressions. For example, the last term in $J_1(\mathbf{p}_1, \mathbf{p}_2)$ gives in the coordinate representation

$$-\frac{Z\alpha^2}{2^5\pi^6}\int d\mathbf{p}_1 d\mathbf{p}_2 \exp(i\mathbf{p}_1\mathbf{r}_1 - i\mathbf{p}_2\mathbf{r}_2)$$

$$\times \int_0^\infty \frac{\xi d\xi dt}{(\xi + t + 1)^2} \frac{\hat{p}_2 \hat{p}_1}{(p_1^2 + p_2^2 t)\xi + (\mathbf{p}_1 - \mathbf{p}_2)^2 t}$$

$$= -\frac{Z\alpha^2}{(2\pi)^2}(\alpha, \alpha\nabla_2)(\nabla_1)\frac{1}{r_{12}(r_1 + r_2 + r_{12})};$$

$$r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|, \quad \nabla_{1,2} = \partial/\partial\mathbf{r}_{1,2}.$$

Consequently the important region in the integration of the expression $\int \mathfrak{M}(\mathbf{r}_1, \mathbf{r}_2)\psi_c(\mathbf{r}_2)d\mathbf{r}_2$ over $\mathbf{r}_2$ is that in which $|\mathbf{r}_1 - \mathbf{r}_2| \lesssim r_1$. In the nonrelativistic case $r_1 > 1/m$ the important region is that in which $|\mathbf{r}_1 - \mathbf{r}_2| \lesssim 1/m$.[6] The difference is due to the difference we have mentioned between the natures of the fluctuation motion for $r > 1/m$ and $r < 1/m$: for $r < 1/m$ the "radius of fluctuation" is proportional to r.

4. In order to find the change of the Coulomb singularity of the S and $P_{1/2}$ states, in what follows we shall seek to obtain directly the quantity $\int \mathfrak{M}(\mathbf{r}_1, \mathbf{r}_2)\psi_c(\mathbf{r}_2)d\mathbf{r}_2$ in Eq. (8); here $\psi_c(\mathbf{r})$ is the Coulomb function. If we consider elements other than the very lightest, so that $rm \sim r_{nuc}m \ll Z\alpha$, the wave function in which we are interested has the form (for definiteness we shall speak of the S state):

$$\psi_c(\mathbf{r}) = \text{const}\cdot r^{-\gamma}\begin{pmatrix} u_0 \\ (i\gamma/Z\alpha)(\boldsymbol{\sigma}\mathbf{r})u_0/r\end{pmatrix}$$

$$u_0 = \begin{pmatrix}1\\0\end{pmatrix} \text{ or } \begin{pmatrix}0\\1\end{pmatrix} \qquad (15a)$$

or in the momentum representation

$$\psi_c(\mathbf{p}) = p^{\gamma-3}\begin{pmatrix} c_1(Z\alpha)u_0 \\ c_2(Z\alpha)(\boldsymbol{\sigma}\mathbf{p})u_0/p\end{pmatrix}, \qquad (15b)$$

where $\gamma = 1 - (1 - Z^2\alpha^2)^{1/2}$, and c_1 and c_2 are certain functions of $Z\alpha$.

Since $\psi_c(\mathbf{p})$ and $\mathfrak{M}$ are homogeneous functions of $\mathbf{p}_1$, $\mathbf{p}_2$, the quantity $\int \mathfrak{M}(\mathbf{p}_1, \mathbf{p}_2)\psi_c(\mathbf{p}_2)d\mathbf{p}_2$ is obviously proportional to $p_1^{\gamma-2}$; that is, in the coordinate representation

$$\int \mathfrak{M}(\mathbf{r}_1, \mathbf{r}_2)\psi_c(\mathbf{r}_2)d\mathbf{r}_2 = \frac{\alpha}{2\pi}\hat{B}\left(Z\alpha, \frac{\mathbf{r}_1}{r_1}\right)\frac{\psi_c(\mathbf{r}_1)}{r_1}, \qquad (16)$$

where $\hat{B}$ is a certain spinor operator; the factor $\alpha/2\pi$ is separated out for convenience. From what has been said there follows the assertion used earlier, that for $p_1 \gg m$ values $p_2 \sim p_1 \gg m$ are also important in $\mathfrak{M}(\mathbf{p}_1, \mathbf{p}_2)$: the integral $\int \mathfrak{M}(\mathbf{p}_1, \mathbf{p}_2)\psi_c(\mathbf{p}_2)d\mathbf{p}_2$ converges in the region $p_2 \sim p_1$.

From the invariance of the complete equation (3), (8) under rotational and space-time reflections it follows that the operator $\hat{B}$ in Eq. (16) has the form

$$\hat{B}\left(Z\alpha, \frac{\mathbf{r}}{r}\right) = Z\alpha f_1(Z^2\alpha^2) + i\frac{\alpha\mathbf{r}}{r}Z^2\alpha^2 f_2(Z^2\alpha^2), \qquad (17)$$

where f_1, f_2 are real even functions of $Z\alpha$.

We shall calculate the operator $\hat{B}$ to accuracy $Z^2\alpha^2$. Using the fact that the difference between each component of the wave function and the first term of its expansion in powers of $Z\alpha$ is a quantity $\sim\gamma \sim Z^2\alpha^2$, and that the operator $\mathfrak{M}$ already contains $Z\alpha$, in finding B we can replace the wave function (15) in the mass integral by the first terms of the expansion:

$$\psi_c(\mathbf{r}) \to \psi_0(\mathbf{r}) \equiv \begin{pmatrix}\varphi_0(\mathbf{r})\\ \chi_0(\mathbf{r})\end{pmatrix} = \text{const}\cdot\begin{pmatrix} u_0 \\ (i\gamma/Z\alpha)(\boldsymbol{\sigma}\mathbf{r}\, u_0)/r\end{pmatrix},$$

$$\psi_c(\mathbf{p}) \to \psi_0(\mathbf{p}) = \text{const}\cdot\begin{pmatrix}\delta(\mathbf{p})u_0 \\ (\gamma/Z\alpha)(\boldsymbol{\sigma}\mathbf{p}\, u_0)/\pi^2 p^4\end{pmatrix}. \qquad (18)$$

Having so obtained the answer in the form of Eq. (16) with ψ as given in Eq. (18), we shall determine the first terms of the expansion of B, i.e., of the functions f_1 and f_2 in Eq. (17). Having found B, we can again use Eq. (16) with the exact Coulomb function $\psi_c(\mathbf{r})$. This decidedly simplifies the calculations; to use the formulas (15) in finding B would be to get useless further accuracy, since B is subject to expansion in powers of $Z\alpha$.

Thus Eq. (8) with $\mathfrak{M}$ from Eq. (14) is now written

$$\sigma p\chi - (Z\alpha/r + V_{\mathbf{pol}})\varphi + m\varphi + \int \mathfrak{M}_1(\mathbf{r}, \mathbf{r}')\varphi_0(\mathbf{r}')d\mathbf{r}' = 0, \qquad (19a)$$

$$\sigma p\varphi - (Z\alpha/r + V_{\mathbf{pol}})\chi - m\chi + \int \mathfrak{M}_1(\mathbf{r}, \mathbf{r}')\chi_0(\mathbf{r}')d\mathbf{r}' = 0. \qquad (19b)$$

The mass term in Eq. (19a) is found immediately:

$$\text{const}\cdot\frac{Z\alpha^2}{2\pi}\frac{1}{2\pi^2}\int\frac{d\mathbf{p}_1 d\mathbf{p}_2\exp(i\mathbf{p}_1\mathbf{r})}{(\mathbf{p}_1 - \mathbf{p}_2)^2}\left(\ln\frac{|\mathbf{p}_1 - \mathbf{p}_2|}{p_1} + \frac{1}{2}\right.$$

$$\left. + \int_0^\infty \frac{d\xi dt}{(\xi + t + 1)^2}\frac{{}^1\!/_2(p_1^2 + p_2^2 t) - \hat{p}_2\hat{p}_1\xi}{(p_1^2 + p_2^2 t)\xi + (\mathbf{p}_1 - \mathbf{p}_2)^2 t}\right)$$

$$\times\delta(\mathbf{p}_2) = \frac{Z\alpha^2}{2\pi r}\varphi_0(\mathbf{r}). \qquad (20)$$

The direct calculation of the mass term in Eq. (19b) is rather cumbersome. Using Eqs. (17) and (20), however, we at once conclude that this term is $(Z\alpha^2/2\pi r)\chi_0(\mathbf{r})$.

Thus, combining the results of the first approximation, we have for the equation (8), with accuracy $Z\alpha^2$

$$(\boldsymbol{\sigma}\mathbf{p} + \beta m - Z\alpha/r - V_{\mathbf{pol}} + Z\alpha^2/2\pi r)\psi(\mathbf{r}) = 0.$$

5. In finding the next approximation we again use Eq. (17); we shall carry through the calculation only for the integral $\int \mathfrak{M}^{(2)}(\mathbf{p}, \mathbf{p}')(2\pi)^3\delta(\mathbf{p}')d\mathbf{p}'$.

According to Eq. (9) we have

$$(2\pi)^3 \mathfrak{M}^{(2)}(\mathbf{p}, 0) = \frac{Z^2\alpha^3}{2^4\pi^7}\beta$$

$$\times \int \frac{d^4k}{k^2}\gamma_\mu \frac{\hat{p}-\hat{k}}{(p-k)^2}\beta\frac{\hat{q}-\hat{k}}{(q-k)^2}\beta\frac{-\hat{k}}{k^2}\gamma_\mu\frac{d\mathbf{q}}{q^2(\mathbf{p}-\mathbf{q})^2} \quad (21)$$

(throughout, the two-dimensional part of the product of the four-rowed matrices is to be understood). It is convenient to separate the factor $-\hat{k} = -\boldsymbol{\gamma}\mathbf{k} - \beta k_4$ into its space and time parts and calculate the corresponding integrals J_A and J_B separately:

$$(2\pi)^3 \mathfrak{M}^{(2)}(\mathbf{p}, 0) = \frac{Z^2\alpha^3}{2\pi}\beta(J_A + J_B),$$

$$J_{A,B} = \frac{1}{2^3\pi^6}\int \frac{d\mathbf{q}}{q^2(\mathbf{p}-\mathbf{q})^2}\gamma_\mu\gamma_n\beta\gamma_m\beta$$

$$\times \int \frac{d^4k}{k^4}(-\boldsymbol{\gamma}\mathbf{k}, -\beta k_4)\gamma_\mu\frac{(p-k)_n(q-k)_m}{(p-k)^2(q-k)^2}. \quad (22)$$

Applying the method that has been described for the integration over k, using the equation $\partial f/\partial \mathbf{k} \times (\mathbf{p}-\mathbf{k}) = -\partial f(\mathbf{p}-\mathbf{k})/\partial\mathbf{p}$ in the calculation of J_A, and noting in the calculation of J_B that the fourth components of $\mathbf{p}$ and $\mathbf{q}$ are absent, i.e., the terms linear in k_4 are to be set equal to zero, we get

$$J_A = -\frac{i\gamma}{(2\pi)^4}\int\frac{d\mathbf{q}}{q^2}\left(\frac{\partial}{\partial\mathbf{q}}+\frac{\partial}{\partial\mathbf{p}}\right)\frac{1}{(\mathbf{p}-\mathbf{q})^2}$$

$$\times\int_0^\infty \frac{d\xi dt}{(\xi+t+1)^2}\,\frac{{}^1\!/_2(p^2+q^2t)-\xi\hat{q}\hat{p}}{(p^2+q^2t)\xi+(\mathbf{p}-\mathbf{q})^2t},$$

$$J_B = \frac{i}{2^3\pi^4}\int\frac{d\mathbf{q}}{q^2(\mathbf{p}-\mathbf{q})^2}\int_0^\infty\frac{\xi d\xi dt}{(\xi+t+1)^2}\left(2\frac{\hat{p}+\hat{q}t}{\xi+t+1}\right.$$

$$\left.-\hat{p}-\hat{q}\right)\frac{1}{(p^2+q^2t)\xi+(\mathbf{p}-\mathbf{q})^2t}. \quad (23)$$

The calculation of the integrals J_A, J_B is accomplished by integration by parts, change of variables, and integration in the complex plane; because of the special nature of the expressions we omit the calculations. The results are:

$$\beta J_A = \frac{\boldsymbol{\alpha}\mathbf{p}}{4\pi p^3}\left(\frac{\pi^2}{12}-\frac{1}{4}\right),\quad \beta J_B = \frac{\boldsymbol{\alpha}\mathbf{p}}{4\pi p^3}\left(\frac{\pi^2}{12}-\frac{5}{4}\right). \quad (24)$$

Equating the Fourier transform of Eq. (17) to Eqs. (24), (22), we find

$$f_2(Z^2\alpha^2) = \tfrac{1}{6}\pi^2 - \tfrac{3}{2} + O(Z^2\alpha^2). \quad (24')$$

The expansion of the vacuum-polarization potential contains only odd powers of $Z\alpha$,[4] so that to our present accuracy Eq. (8) for small distances can be written

$$\Big[\boldsymbol{\alpha}\mathbf{p} + \beta m - \frac{Z\alpha}{r} - V_{\text{pol}}$$

$$+\frac{Z\alpha^2}{2\pi r} + \frac{i\boldsymbol{\alpha}\mathbf{r}}{r^2}\frac{Z^2\alpha^3}{2\pi}\left(\frac{\pi^2}{6}-\frac{3}{2}\right)\Big]\psi(\mathbf{r}) = 0. \quad (25)$$

For the main component $\varphi(\mathbf{r})$ of the wave function we get in the region in question

$$\varphi(\mathbf{r})/\varphi_c(\mathbf{r}) = 1 + \delta\varphi_{\text{pol}}/\varphi_c - \delta\varphi_{\text{fl}}/\varphi_c$$

$$= 1 + \frac{Z^2\alpha^3}{2\pi}\left\{\frac{2}{3}\left[\ln^2\frac{1}{rm} + \ln\frac{1}{rm} + \frac{1}{2}\right.\right.$$

$$\left.\left.-\left(\frac{5}{6}+\ln\gamma\right)\ln\frac{1}{mZ\alpha r}\right] - \left(\frac{\pi^2}{6}-\frac{1}{2}\right)\ln\frac{1}{mZ\alpha r}\right\}. \quad (26)$$

For $Z = 92$, $r \sim r_{\text{nuc}} = 6\times10^{-13}$ cm the right member of Eq. (26) differs from unity by

$$\delta\varphi(r_{\text{nuc}})/\varphi_c(r_{\text{nuc}}) = 2.7\cdot10^{-3}. \quad (27)$$

At present such a small change of the wave function cannot be distinguished from the effects of the finite dimensions of the nucleus, the distribution of charge in the nucleus, etc., and is obviously unobservable.

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