where the index i refers to the break-up products; v is the speed of the initial particle (Z_a, a); b is its impact parameter relative to the target nucleus, and $\mu_a = aAM_p/(A+a)$ is their reduced mass. In such a case the cross section of the examined process obviously will be equal to

$$\sigma = \pi b_{max}^2, \tag{2}$$

where b_{max} is defined from the condition that the acquired energy of relative motion E′ of the break-up products will be equal to (or greater than) the binding energy ϵ_0 of this system, that is

$$E' = \frac{1}{2\mu}\left(\frac{a_1}{a}\mathbf{p}_2 - \frac{a_2}{a}\mathbf{p}_1\right)^2 \geqslant \varepsilon_0, \qquad \mu = \frac{a_1 a_2}{a} M_p. \tag{3}$$

By substituting here p_1 and p_2 from formula (1), and solving the resulting equation for b^2 and then averaging it over all emission angles, we obtain the following expression for the cross section after some simple transformations:

$$\sigma = \frac{\pi(Ze^2)^2}{E_0\varepsilon_0}\,\frac{Z_1^2a_2^2 + Z_2^2a_1^2}{a_1a_2}\left(1 - \frac{\varepsilon_0}{E_0}\,\frac{Z_a^2a_1a_2}{Z_1^2a_2^2 + Z_2^2a_1^2}\left(\frac{A+a}{2A}\right)^2\right). \tag{4}$$

Here E_0 is the kinetic energy of the incident particle, which for relativistic speeds must be replaced in formula (4) by the quantity $E_0(1+E_0/2M_ac^2)(1+E_0/M_ac^2)^{-2}$. The formula obtained for the cross section is similar to the Dancoff results calculated by numerical integration for $E_0 \sim 200$ Mev and $A \sim 100$.

Since we are interested in the case $E_0 \gg \epsilon_0$ then, from relation (4) we find (expressing E_0 and ϵ_0 in Mev),

$$\sigma \approx 6.3 \cdot 10^{-26}\,\frac{Z^2}{\varepsilon_0 E_0}\,\frac{a_2^2Z_1^2 + a_1^2Z_2^2}{a_1a_2}\,\text{cm}^2. \tag{5}$$

As can be seen from the resulting formulas, the estimate of cross sections for the processes under consideration does not present any particular difficulty. Thus for instance, for deuterons ($\epsilon_0 = 2.18$) with an energy of ~ 200 Mev and for beryllium ($\epsilon_0 = 1.7$) with an energy of ~ 100 Mev, we have $1.4 \times 10^{-28} Z^2$ and $0.45 \times 10^{-28} Z^2$ cm² respectively. These magnitudes differ somewhat from those calculated in references 1 and 2 but the latter are also approximations and have not been checked by experiment.

In conclusion we note that the same formulas can be used to estimate the cross section of the Coulomb break-up of the molecules H_2^+, D_2^+ and others (supposing that $Z_1 \sim 1$; $Z_2 \sim 0$; $a_1 = a_2$; $\epsilon_0 \sim 2.5$ ev), for which with energies of 20 — 30 kev we obtain $\sigma \approx 10^{-18} Z^2$ cm². The author is grateful to Yu. V. Kursanov who pointed out the possibility of such processes taking place in ion sources with a high degree of ionization.

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A SIMPLE METHOD OF CALCULATING THE DEGREE OF IONIZATION AND THERMODYNAMIC FUNCTIONS OF A MULTIPLY IONIZED IDEAL GAS

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The thermodynamic functions of a gas at high temperatures, when the gas atoms are multiply ionized, are calculated on the basis of ionization equilibrium. For each pair of values of the temperature and density we must solve a nonlinear system consisting of a few nonlinear algebraic equations for ion concentrations; this requires a long calculation. Such calculations have thus far been carried out only for air[1] from 20,000° to 500,000° and from 10 to 10^{-3} times normal pressure. We here suggest a simple method for obtaining a fairly accurate estimate of the degree of ionization and of the thermodynamic functions of any gas.

We write a set of Saha equations for a gas consisting of atoms of a single element:

$$N_eN_{n+1}/N_n = (2g_{n+1}/g_n)(2\pi m_e kT/h^2)^{3/2}\exp(-I_{n+1}/kT),$$

$$N = \Sigma N_n, \quad N_e = \Sigma nN_n; \qquad n = 0,\ 1,\ 2\ldots Z. \tag{1}$$

Here N, N_n, N_e are the numbers of the original atoms, n-multiply ionized atoms and electrons per cm³, and I_{n+1} is the (n+1)-th ionization potential. The statistical weight ratio g_{n+1}/g_n for the electronic states of the ions depends on the temperature and varies irregularly with n and with a change from one element to another. However, since it remains of the order of unity we shall assume its value to be exactly 1.

We regard the ion numbers N_n and ionization potentials I_n as continuous functions of n. We construct I(n) graphically by drawing a continuous curve through the discrete values of I_n obtained from tables,[2] with $I_0 = I(0) = 0$. The system of algebraic equations is transformed into a differential equation by the assumption $N(n+1) = N(n) + dN/dn$. For each pair of temperature and density values the gas contains an appreciable number of ions of two or at most three types; the distribution function N(n) is thus a narrow peak. For the mean ionic charge $\bar{n} = N_e/N$ we shall take the value of n at the maximum of N(n). This becomes increasingly valid as the peak of N(n) becomes narrower. Under these assumptions we obtain from (1) a transcendental equation for determining the degree of ionization or mean ionic charge multiplicity $\bar{n}$ as a function of temperature and density; after taking the logarithm we obtain*

$$I(\bar{n} + {}^1/_2) = kT \ln (AT^{3/2}/N\bar{n}), \quad A = 2(2\pi m_e k/h^2)^{3/2}. \tag{2}$$

Because of the logarithmic dependence of the right-hand side on $\bar{n}$ two or three successive approximations are sufficient to obtain a very exact value of the root $\bar{n}$ with the aid of the graph of I(n).

Approximate formulas for the thermodynamic functions are obtained from the exact formulas when, in accordance with the given simplifications, we assume the distribution of N_n, or rather N(n), to be a δ function around $\bar{n}$. The internal energy per original atom and the pressure are then given by

$$\varepsilon = \frac{3}{2}\left(1 + \frac{N_e}{N}\right)kT + \frac{1}{N}\Sigma N_n Q_n \approx \frac{3}{2}(1+\bar{n})\,kT + Q(\bar{n}); \tag{3}$$

$$p = (N + N_e)\,kT \approx N(1+\bar{n})kT, \tag{4}$$

where Q_n is the energy required for the successive stripping of the first n electrons from a neutral atom and varies along a continuous curve Q(n) plotted through discrete values of Q_n. Electron excitation energy, which is usually small, is here disregarded.

The entropy per original atom is given by

$$S = k\Sigma \frac{N_n}{N} \ln \left(\frac{2\pi MkT}{h^2}\right)^{3/2} \frac{g_n e^{5/2}}{N_n} + k\frac{N_e}{N} \ln \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \frac{2 e^{5/2}}{N_e}$$

$$\approx k \ln \left(\frac{2\pi MkT}{h^2}\right)^{3/2} \frac{e^{5/2}}{N} + k\bar{n}\ln \frac{AT^{3/2} e^{5/2}}{N\bar{n}}. \tag{5}$$

The electronic statistical weight g(n) of the ions is set equal to unity. When S = const the resulting equation

$$\frac{T^{3/2}}{N} \exp\left\{\bar{n}\left(\frac{I(\bar{n} + {}^1/_2)}{kT} + \frac{5}{2}\right)\right\} = \text{const} \tag{6}$$

combined with (2) represents the equation of the adiabate $T = T(N, T_0, N_0)$ in parametric form (with $\bar{n}$ as the parameter).

The same method can easily be applied to a gas consisting of a mixture of two or more elements.

The accuracy of the method is illustrated by the table, which contains the results of our computations, for air, of the average number of particles per atom $(1+\bar{n})$ and the internal energy per atom in ev (these are the upper values), compared with the results obtained by Selivanov and Shlyapintokh (the lower values) at a few temperatures and densities. For simplicity the gas was assumed to consist of a single element and the ionization potentials I_n were averages according to the nitrogen-to-oxygen ratio of the air. The molecular dissociation energy was added to the internal energy. The table shows entirely satisfactory agreement even for low degrees of ionization (when the largest errors should result).

T, °K	$1+\bar{n}$	ε	$1+\bar{n}$	ε	$1+\bar{n}$	ε
	$\rho/\rho_{\mathrm{norm}}=1$		10^{-1}		10^{-2}	
30000	1.68	16.6	1.97	21.6	2.3	33
	1.77	23	2.04	27.6	2.21	33
50000	2.4	40.5	2.85	58.5	3.35	83
	2.42	47.8	2.85	64	3.26	80
100000	3.72	126	4.47	186	5.1	243
	3.75	140	4.45	190	5.16	252

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*The ionization potential in the Saha equation (1) bears the index n + 1 with $I_0 = 0$. Often I_n is written, in which case I_0 is the first ionization potential; this is merely a matter of notation. When we go over to the continuous functions N(n) and I(n) the value of n depends on the choice of notation. A comparison with the exact calculation shows that the mean potential pertains accurately to the middle of the interval $\bar{n}$, $\bar{n} + 1$ and the I(n) curve is connected to the point I(0) = 0.

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