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ON THE SHAPE OF EVEN-EVEN NUCLEI

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A nuclear model assuming a core plus two nucleons in a shell with angular momentum j is considered. The energy is determined as a function of the parameters β and γ for various values of the total angular momentum of the nucleons. It is shown that the minimum energy in the ground state corresponds to a shape of the nucleus without axial symmetry, provided that $j > 3/2$.

INTRODUCTION

FOR a long time physicists were convinced that the atomic nuclei have spherical shape. The success of the liquid drop model of the nucleus strengthened this belief. However, the detailed study of the ground and excited states of the nucleus established the fact that many nuclei deviate from the spherical shape. In particular, this holds for nuclei whose mass numbers lie in the regions $A > 225$, $155 < A < 185$, and $A \sim 24$.

The nonspherical shape of the nucleus manifested itself in the presence of the rotational spectrum for the excited states, and in the large electric quadrupole moments of the stationary states of the nucleus, in the measurement of which great progress has been made thanks to the method of nuclear Coulomb excitation and the investigation of the γ transitions in nuclei.

The sizable deviation of the equilibrium shape of the nucleus from spherical symmetry remained unexplained for a long time. The first interpretation was given by J. Rainwater[1] based on a study of the interaction of the nuclear surface with the outer nucleons, i.e., the nucleons which do not belong to completely filled shells. However, it is assumed in this and many later papers[2-5] that the nucleus preserves its axial symmetry. Formally, this amounts to neglecting those parts of the Hamiltonian which are not diagonal in the quantum numbers of the projection of the angular momentum on one of the nuclear axes. The energy of the interaction of the outer nucleons with the nuclear surface therefore was averaged in effect only over nucleon states with a definite value for the angular momentum projection on this axis.

It was shown recently[6] that many properties of the first excited states of even-even nuclei (the order of succession of the spins of the excited states, their energies, and the probabilities for electromagnetic transitions between them) can be readily explained by assuming that the equilibrium shape of the nucleus can in first approximation be represented by a three-axial ellipsoid. The nuclear ellipsoid of Bohr is characterized by the two parameters β and γ; the relations

$$a_0 = \beta\cos\gamma, \qquad a_1 = a_{-1} = 0, \qquad a_2 = a_{-2} = (\beta/\sqrt{2})\sin\gamma$$

connect these parameters with the parameters a_μ defining the shape of the nucleus:

$$R(\vartheta, \varphi) = R_0 + R_0 \sum_{\mu=-2}^{2} a_\mu Y_{2\mu}(\vartheta, \varphi),$$

in the coordinate system attached to the nucleus. Varying the "asymmetry" parameter γ from 0 to $\pi/3$, with a fixed value β, induces a change of the nuclear shape from a prolate to an oblate elliposid of revolution. The value $\gamma = 30°$ corresponds to a shape which is intermediate between the prolate and the oblate ellipsoids of revolution. In order to obtain agreement with experiment, it had to be assumed in reference 6 that in some nuclei the equilibrium value of γ can reach values close to 30°. This large deviation from axial symmetry calls for a theoretical justification.

The first indications of the possibility that the equilibrium shape of the nuclei may deviate from axial symmetry came from the calculations of Gursky,[7] the results of which were quoted in the paper of Wilets and Jean.[8] These calculations showed that the minimal energy of a nucleon system consisting of 55 protons and 91 neutrons moving in the field of a three-axial ellipsoid corresponds to the values $\beta = 0.04$ and $\gamma = 7.5$. Similar calculations in the same approximation were carried out by Geĭlikman[9] for a three-dimensional oscillator potential and by Zaikin[10] for a square-well potential. In these papers it is also shown that the minimum energy of the nucleon system corresponds, in a number of cases, to a nuclear shape without axial symmetry. Unfortunately, these numerical estimates lose in value owing to the fact that the spin-orbit interaction is neglected and a special form of the potential is chosen.

In the present paper we propose a new method for the explanation of possible deviations of the equilibrium shape of the nucleus from axial symmetry, which is based on a generalization (to nuclei without axial symmetry) of the method of Bohr.[2]

1. FORMULATION OF THE PROBLEM

We consider a system consisting of a certain number of nucleons forming the core of the nucleus plus two equivalent outer nucleons in a shell with the definite angular momentum j. According to the Pauli principle the total angular momentum of the two nucleons can take only even values:

$$J = 0,\ 2, \ldots\ 2j-1. \tag{1.1}$$

If, in the zeroth approximation, the coupling with the nuclear surface deformation is neglected (the nucleons move in the field of the nuclear core), the total angular momentum of the nucleon J and its projection M on the ζ axis of the coordinate system $\xi\eta\zeta$ attached to the nucleus are integrals of the motion, whose energy is determined by the operator H_p. States ψ_{JM} differing in the values of J and M belong to the same energy, i.e.,

$$(H_p - E_p)\psi_{JM} = 0. \tag{1.2}$$

In accordance with the unified model of Bohr and Mottelson we assume further that the properties of the nuclear core are determined by the operators of the collective motion

$$H_r = \frac{1}{2}\left\{a(I_\xi - J_\xi)^2 + b(I_\eta - J_\eta)^2 + c(I_\zeta - J_\zeta)^2\right\}, \tag{1.3}$$

$$H_{vib} = -\frac{\hbar^2}{2B}\left\{\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\left(\beta^4\frac{\partial}{\partial\beta}\right) + \frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\left(\sin 3\gamma\frac{\partial}{\partial\gamma}\right)\right\} + \frac{C}{2}\beta^2, \tag{1.4}$$

where

$$a = \hbar^2[4B\beta^2\sin^2(\gamma - 2\pi/3)]^{-1},$$

$$b = \hbar^2[4B\beta^2\sin^2(\gamma + 2\pi/3)]^{-1}, \quad c = \hbar^2[4B\beta^2\sin^2\gamma]^{-1}, \tag{1.5}$$

C is the elastic constant of the nuclear surface, B is the mass parameter, I_ξ, I_η, I_ζ are the projections of the total angular momentum of the nucleus, and J_ξ, J_η, J_ζ are the projections of the total angular momentum of the outer nucleons.

We assume that the pair of outer nucleons interacts with the core as a whole. Then the operator corresponding to this interaction can be written in the form

$$H_{int} = T\beta\left[\cos\gamma(3J_\zeta^2 - J^2) + \sqrt{3}\sin\gamma(J_\xi^2 - J_\eta^2)\right], \tag{1.6}$$

where T is a parameter which determines the strength of the coupling between the nucleon pair in the j shell and the nuclear surface.

The assumptions at the basis of the expression (1.6) require, of course, a detailed justification. They represent a different limiting case from that considered in the paper of Ford,[5] where it is assumed that the outer nucleons interact independently with the nuclear surface. The H_{int} introduced by Ford completely ignores the interaction between the outer nucleons. Ford's Hamiltonian can therefore not be used in the study of the effects connected with the pairing of the nucleons. In the present paper we shall postulate the interaction (1.6). Moreover, we shall not neglect (as was done in references 2 to 5) the terms in the Hamiltonian which give rise to nondiagonal matrix elements with respect to the magnetic quantum numbers, i.e., we shall not assume that $j_3 = \Omega$ is a good quantum number. Therefore we shall not restrict the class of admissible nucleon states to only those states for which

$$\sum_i \Omega_i = 0. \tag{1.7}$$

In those papers in which only states satisfying the subsidiary condition (1.7) are considered, the equilibrium shape of the nucleus will, of course, always turn out to be axially symmetric.

If the motion connected with the changes in β and γ is slow in comparison with the motion of the outer nucleons, we can apply the adiabatic approximation (which was also used in references 2 to 5). Thus we calculate the energy of the whole system for fixed but arbitrary values β and γ, and determine the values β_0 and γ_0 for which the energy becomes a minimum. These values, then, will also determine the equilibrium shape of the nucleus.

In the adiabatic approximation we can neglect in (1.4) the operators corresponding to the kinetic energy. Then the operator for the surface oscillations takes the form

$$H_{vib}^a = {}^1/_2 C\beta^2. \tag{1.4a}$$

The total angular momentum of an even-even nucleus in the ground state is equal to zero. The operator for the rotation energy (1.3) in the ground state can therefore be written in the form

$$H_r^0 = {}^1/_2 \{ aJ_\xi^2 + bJ_\eta^2 + cJ_\zeta^2 \}. \qquad (1.3a)$$

The total Hamiltonian for the ground state of the nucleus in the adiabatic approximation then has the form

$$H = H_p + H_r^0 + H_{vib}^a + H_{int}. \qquad (1.8)$$

In the next section we shall find the solution to the equation

$$[H - E(\beta, \gamma)]\psi = 0, \qquad (1.9)$$

which determines the energy of the system as a function of β and γ.

2. CALCULATION OF THE NUCLEAR ENERGY AS A FUNCTION OF β AND γ

For the determination of the energy $E(\beta, \gamma)$ in Eq. (1.9) we write the wave function ψ in the form

$$\psi = \sum_{JM} a_{JM} | JM \rangle,$$

$$| JM \rangle = [(1 + \delta_{MO})2]^{-1/2} (\psi_{JM} + \psi_{J,-M}), \qquad (2.1)$$

where J runs through the values (1.1), and M = 0, 2, 4, ..., J. Substituting (2.1) in (1.9), we obtain a system of equations for the determination of the coefficients a_{JM}. The secular equation for this system determines $E(\beta, \gamma)$.

The non-zero matrix elements of the operators (1.3a) and (1.6) are

$$\langle JM | H_r^0 | JM \rangle = \frac{a+b}{4} \{J(J+1) - M^2\} + \frac{c}{2} M^2,$$

$$\langle J, M+2 | H_r^0 | JM \rangle = \langle JM | H_r^0 | JM+2 \rangle = \frac{a-b}{8} F(JM),$$

$$\langle JM | H_{int} | JM \rangle = T\beta\cos\gamma\, [3M^2 - J(J+1)],$$

$$\langle JM | H_{int} | JM+2 \rangle = \frac{\sqrt{3}}{2} T\beta F(JM) \sin\gamma, \qquad (2.2)$$

where

$$F(JM) = \{(1 + \delta_{MO})(J - M)$$
$$\times (J - M - 1)(J + M + 1)(J + M + 2)\}^{1/2},$$

From this we see that J is an integral of the motion, so that the equation for $E(\beta, \gamma)$ splits up into a number of simpler equations for each value J.

For J = 0 (pairing of nucleons with opposite momenta), the interaction between the surface deformation and the nucleon pair is zero in our approximation. In this case the nuclear energy depends on β and γ only through E_p and the potential energy of deformation. Since $C\beta^2/2$ does not depend on γ, but E_p obviously is weakly γ dependent, we shall simply neglect the dependence of E_p on γ in the investigation of the dependence of the nuclear energy on γ. Introducing $\epsilon = E(J) - E(0)$, where $E(0)$ is the nuclear energy for J = 0, we consider the equation for the determination of ϵ for J = 2:

$$\begin{vmatrix} {}^3/_2(a+b) - 6T\beta\cos\gamma - \varepsilon, & 6T\beta\sin\gamma + (a-b)\sqrt{3}/2 \\ 6T\beta\sin\gamma + (a-b)\sqrt{3}/2, & 6T\beta\cos\gamma + (a+b)/2 + 2c - \varepsilon \end{vmatrix} = 0.$$

Substituting (1.5), and expanding the determinant, we obtain the second degree equation

$$x^2 - \frac{9x}{2\sin^2 3\gamma} - \frac{9l^2}{4} - \frac{27\cos 3\gamma}{4\sin^2 3\gamma} l + \frac{9}{2\sin^2 3\gamma} = 0, \qquad (2.3)$$

where

$$x = \varepsilon / (\hbar^2/B\beta^2), \qquad l = 4T\beta / (\hbar^2/B\beta^2), \qquad (2.4)$$

and $\hbar^2/B\beta^2$ is the energy of the first excited level of the nucleus.

For a rough estimate of the quantity l we set $T \approx 40$ Mev, $\hbar^2/B\beta^2 \sim 400$ kev, and $\beta = 0.2$. Then we find from (2.4) $l \approx 80$. As the deformation parameter β increases, the parameter l increases as $\sim \beta^3$.

In Fig. 1 we show the solutions of (2.4) as functions of γ for the values $l = 10$, 15, and 150.

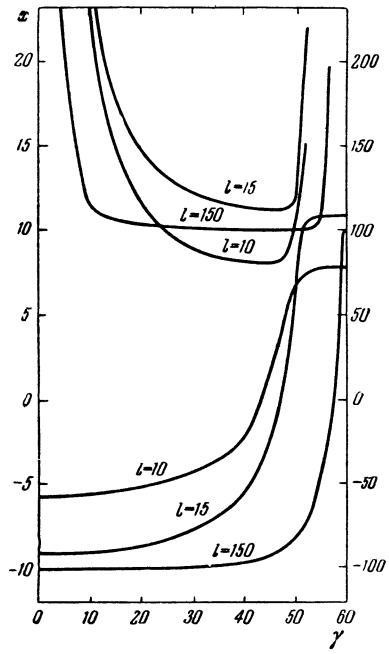

FIG. 1. Nuclear energy as a function of γ and of the quantity l which determines the coupling of the pair of outer nucleons (J = 2) with the nuclear surface deformation. The right hand scale gives the energy for $l = 150$.

In the case when the pair of outer nucleons is in a state with J = 4, the equation for the total nuclear energy has the form

$$x^3 - \frac{45x^2}{2\sin^2 3\gamma} - \left(39l^2 + 117l \frac{\cos 3\gamma}{\sin^2 3\gamma} - \frac{81}{\sin^4 3\gamma} - \frac{78}{\sin^2 3\gamma} \right) x$$
$$- 70 l^3 \cos 3\gamma + 5\left(42 - \frac{9}{2\sin^2 3\gamma}\right) l^2$$
$$+ 5l\left(81 \frac{\cos 3\gamma}{\sin^4 3\gamma} + 42\right) - \frac{270}{\sin^4 3\gamma} - \frac{70}{\sin^2 3\gamma} = 0. \qquad (2.5)$$

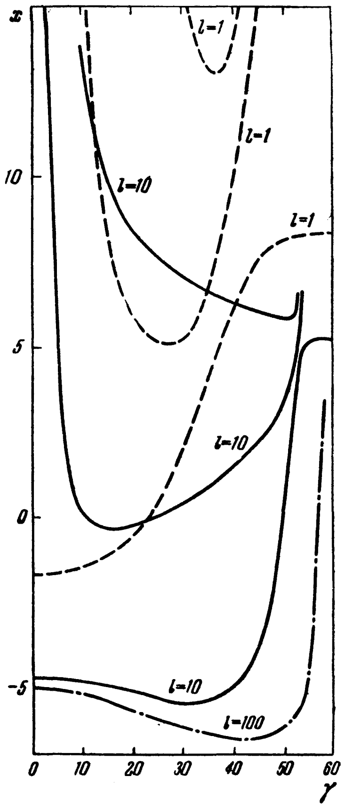

FIG. 2. The energy as a function of γ and l for $J = 4$. The value of the energy is to be multiplied by 10 for $l = 10$, and by 100 for $l = 100$.

where x and l are given by (2.4). Figure 2 shows the dependence of three roots of (2.5) on γ for the value $l = 10$ (solid curve). In the same figure we give the dependence of the lowest roots of (2.5) on γ for $l = 1$ (dotted curve) and $l = 100$ (dash-dotted curve).

3. THE SHAPE OF THE NUCLEUS AS A FUNCTION OF THE STRENGTH OF THE COUPLING BETWEEN THE PARTICLES AND THE NUCLEAR SURFACE

We apply the results of the preceding section to the problem of finding the equilibrium shape of the nucleus.

If each of the outer electrons is in the state $j = 1/2$, they can be paired only with the value $J = 0$. Such a nucleon pair has no effect on the shape of the nucleus.

If the nucleons are in the state $j = 3/2$, pairing can occur for $J = 0$ and $J = 2$. According to Fig. 1, the state with $J = 2$ has the lower energy. The minimum of the energy then corresponds to a nuclear shape with axial symmetry ($\gamma_0 = 0$). A pair of outer nucleons in states with $j = 3/2$ therefore does not destroy the axial symmetry of the nucleus.

If the nucleons are in the state $j = 5/2$, pairing can occur for $J = 0$, 2, and 4. A comparison of Figs. 1 and 2 shows that the minimal energy for a given value l corresponds to the largest possible angular momentum ($J = 4$). In this case the equilibrium shape of the nucleus for $l = 1$ also corresponds to an ellipsoid of revolution ($\gamma_0 = 0$). However, as l increases, the equilibrium shape of the nucleus corresponds to $\gamma_0 \neq 0$. As l reaches the value 10, the equilibrium shape of the nucleus corresponds to $\gamma_0 \approx 30°$. Here the deviation from axial symmetry reaches a maximum: the nucleus has a shape which is intermediate between the prolate and the oblate ellipsoids of revolution. For $l = 100$, $\gamma_0 \approx 42°$; for $l \rightarrow \infty$, $\gamma_0 \rightarrow 60°$, i.e., the shape of the nucleus becomes again axially symmetric. In the case of extremely strong coupling between the nucleons and the surface (β large) our results are therefore the same as those of Bohr and Ford.

If the nucleons in the pair are in a state with $j \geq 7/2$, we continue to observe the same behavior as in the case $j = 5/2$. The minimal energy corresponds to the largest possible value J. As the quantity l increases, the equilibrium value of γ changes from 0 to 60°. And the larger the value of J, the smaller is the value of l at which the deviation from axial symmetry begins.

Thus the effect of the interaction of the nucleon pair with the nuclear surface favors the pairing of the nucleons with the largest possible value of J. This effect therefore acts in the opposite direction of the effect of the attractive interaction of the fermions at the Fermi surface (in momentum space), which in certain cases (superconductivity of metals) leads to a coupling of the fermions with opposite spins and momenta. For sufficiently small l the competition between these effects favors the states with $J = 0$. For large l, on the other hand, the states with the largest possible J are more favored energetically. Owing to the Coulomb repulsion between protons, this occurs at smaller values of l for protons than for neutrons.

If the number of nucleons in the shell of angular momentum j corresponds to a closed shell, this state will have only one total angular momentum: $J = 0$.

Using the formulas of reference 6 and the experimental value for the ratio of the energy of the second excited level with spin 2 over the energy of the first excited level, we can determine the equilibrium value γ_0, up to the transformation $\gamma_0' \rightleftarrows 60 - \gamma_0$. This ambiguity arises from the fact that the position of the levels and the transition probability do not depend on the sign of the quadrupole moment. These calculations lead to the result that the greatest deviation from axial symmetry ($30° > \gamma_0 > 27°$) occurs in the nuclei $_{46}Pd^{108}$, $_{46}Pd^{106}$, $_{48}Cd^{114}$, $_{52}Te^{122}$, $_{52}Te^{126}$, $_{78}Pt^{192}$, $_{78}Pt^{196}$, and several others in which the number of protons differs from the magic numbers 50 and 82 by two or four units. For $T = 40$ Mev and values β calculated from Coulomb excitation data, the parameter l, which determines the dependence of the nuclear ground state energy on γ, lies somewhere within the interval 20 to 80.

For nuclei with a ratio $E(2')/E(2)$ between the limits 2.9 and 20 (which corresponds to values of γ within the intervals $20° > \gamma_0 > 10°$ or $40° < \gamma_0 < 50°$), the parameter l has the value 300 to 600. Finally, for nuclei with nearly axial symmetry, for which $E(2')/E(2) > 23$ ($\gamma_0 < 10°$ or $\gamma_0 > 50°$), the parameter $l > 1000$. These experimental values are in qualitative agreement with the theoretical results.

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293