PROPERTIES OF THE SPECTRUM OF ELEMENTARY EXCITATIONS NEAR THE DISINTEGRATION THRESHOLD OF THE EXCITATIONS

L. P. PITAEVSKIĬ

Institute of Physics Problems, Academy of Sciences, U.S.S.R.

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The singularity of the Bose-liquid Green's function near the excitation disintegration threshold is investigated by quantum-field theory methods without assuming weakness of the interaction. It is shown that three possible types of decay thresholds exist. In the first case the excitation velocity at the threshold point $p = p_c$ equals that of sound, so that starting with this point the excitation can produce phonons, thus leading to damping proportional to $(p_c - p)^3$. In the two other cases, excitation at the threshold can break up into two excitations with non zero momenta, which are either parallel to each other or form a definite angle. In either case, the spectrum curve ends at the threshold point, and the excitation velocity at this point equals that of each of the excitations produced in the decay. Scattering of neutrons in the liquid, involving the production of excitations near the threshold is considered.

1. STATEMENT OF THE PROBLEM. POSSIBLE TYPES OF DISINTEGRATION THRESHOLDS

THIS paper is devoted to an investigation of the properties of the spectrum of elementary excitations in a Bose liquid near its point of termination. As is known, at small momenta, the excitations in a Bose liquid are phonons, i.e., the excitation energy depends linearly on the momentum. As the momentum increases, the spectrum deviates from linear and its further course depends on the specific properties of the interaction between the liquid particles, and cannot be calculated theoretically in general form. Thus, the energy spectrum of liquid He^4 has a complex form with a minimum at $p = 2 \times 10^{-19}$ g-cm/sec. Upon further increase in momentum, the excitation energy reaches a certain threshold value, above which the excitation is unstable and may break up into two or more excitations with lower energies. This point is the end point of the spectrum, and at larger momenta undamped excitations no longer exist.* The end point is a singular point of the spectrum, and the work presented is devoted to a clarification of the character of this singularity.† We shall see later that this investigation can be carried out in general form, without assumption of weak interaction.

*This pertains to absolute zero temperature. At absolute zero, disintegration is the only mechanism of damping Bose excitations.

†This statement of the problem is due to L. D. Landau.[1]

When an excitation breaks up into two, the energy and momentum conservation laws must be satisfied, as given by

$$\varepsilon(p) = \varepsilon(q) + \varepsilon(|\mathbf{p} - \mathbf{q}|). \tag{1}$$

Here $\mathbf{p}$ and $\epsilon(p)$ are the momentum and energy of the excitation that breaks up, $\mathbf{q}$ and $\epsilon(q)$ are the momentum and energy of one of the excitations formed in the breakup, and $\mathbf{q}_1 = \mathbf{p} - \mathbf{q}$ and $\epsilon(|\mathbf{p} - \mathbf{q}|)$ are the momentum and energy of the second excitation produced. If Eq. (1) does not have solutions for $\mathbf{q}$ for a given $\mathbf{p}$, no disintegration is possible. The threshold of the disintegration [we denote the momentum of the excitation at the threshold point by $\mathbf{p}_c$, and the energy by $\epsilon_c = \epsilon(p_c)$] is characterized by the fact that Eq. (1), when taken with respect to $\mathbf{q}$, has no solutions for $\epsilon < \epsilon_c$ and has a solution for $\epsilon = \epsilon_c$. It is obviously necessary for this that the right half of (1), when expressed as a function of $\mathbf{q}$, have a minimum at $p = p_c$ for certain values of $\mathbf{q}$. When $p = p_c$ the right half of (1) depends on two variables, q and $\cos\vartheta$, where ϑ is the angle between the vectors $\mathbf{p}$ and $\mathbf{q}$. For the foregoing expression to have a minimum with respect to ϑ, it is necessary that $\cos\vartheta$ have a maximum, i.e., that the angle ϑ be zero, or else that the function $\epsilon(|\mathbf{p} - \mathbf{q}|) = \epsilon(q_1)$ have a minimum with respect to $\cos\vartheta$ for certain $\vartheta = \vartheta_0$, and consequently, that it have a minimum with respect to its argument q_1 at a certain value $q_1 = p_0$.

In the former case, the excitations produced in the disintegration proceed "forward," i.e., along the direction of the vector $\mathbf{p}$, and have identical velocities v_c. (Otherwise the right half of (1) would contain terms linear in the increments of $\mathbf{q}$ and could not have a minimum). There are two possibilities here. First, one of the disintegration excitations may have a momentum as close to zero as desired. This corresponds to the case when the speed of excitation at the point p_c equals the velocity of sound c, and the excitation may produce a phonon (case a). Second, the two created excitations may have finite momenta (case b).

Let us proceed to the case of finite ϑ. In this case each of the excitations is produced with a momentum equal to p_0, at which the energy $\epsilon(q)$ is a minimum, i.e., it has in the vicinity of p_0 the "roton" form

$$\varepsilon(q) = \Delta + (q - p_0)^2/2\mu. \tag{2}$$

If the excitation energy is $\epsilon = \epsilon_c = 2\Delta$, the excitation breaks up into two rotons with $q = p_0$ and $\epsilon = \Delta$, emitted at an angle ϑ_0 such that the sum of the momenta is p_c (case c); it is naturally necessary for this that $p_c < 2p_0$. These three cases cover all types of thresholds of disintegration into two excitations.*

To investigate the form of the spectrum near the threshold we use the methods of quantum field theory. Specifically, we calculate the Green's function of the elementary excitations, the poles of which indeed determine the excitation spectrum. We consider here elementary excitations as being Bose particles capable of disintegrating. We also assume that the interaction between the excitations has a three-particle character, i.e., that the interaction Hamiltonian is in the form of a product of three ψ-operators. This assumption is made only to simplify the formulas. It is easy to verify that the result is not changed if the Hamiltonian contains terms with products of a larger number of operators. Let us note that the Bose liquid can be considered more rigorously by the technique developed by Belyaev,[4] in which the system is described not by one but by two Green's functions. It can be shown, however, that in our case both approaches lead to the same results. This is caused by the fact that both functions, as can be readily shown in the general case, have the same poles, so that any modification of the analysis given below reduces to the appearance of equations with not one but several terms having an identical analytical behavior.

*N. N. Bogolyubov[2] has shown that a nearly-ideal Bose gas would have a phonon spectrum with $(\partial^2\varepsilon/\partial p^2)_{p=0} > 0$. Such a spectrum would be unstable from the very outset. According to calculations by Belyaev,[3] the attenuation damping in this case is proportional to p^5 at small p. In real helium, however, $(\partial^2\varepsilon/\partial p^2)_{p=0} < 0$.

2. FORM OF THE SPECTRUM NEAR THE THRESHOLD POINT

To clarify the character of the singularity of the Green's function $G(p)$, we use the Dyson equation, the form of which is shown in Fig. 1.[5]

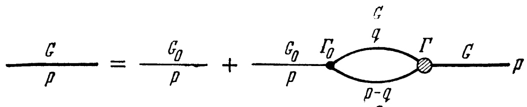

FIG. 1

The heavy line denotes here the complete Green's function G, and the thin line the free Green's function G_0, while the shaded circle denotes the complete vertex part Γ and the dot denotes the vertex point Γ_0 in the perturbation-theory approximation. In analytical form this expression becomes

$$G^{-1}(p) - G_0^{-1}(p) = i\int \Gamma(p; q, p-q)$$
$$\times G(q)\,G(p-q)\,\Gamma_0(p; q, p-q)\,d^4q/(2\pi)^4. \tag{3}$$

Hereinafter p in the arguments of Green's functions and in the vertex parts will denote the energy-momentum 4-vector with components $\{\epsilon, \mathbf{p}\}$, and analogously $q = \{\omega, \mathbf{q}\}$ etc.

Near the thresholds of all three types listed in Sec. 1, the equations in (3) have properties that are entirely different, and we must consider each case separately.

a. Properties of the spectrum near the threshold of phonon creation. We consider the properties of the excitation spectrum near the point where the speed of excitation becomes equal to the speed of sound. Starting with this point, the excitation may create a phonon. The conservation law (1) becomes in this case

$$\varepsilon(p) = \varepsilon(|\mathbf{p} - \mathbf{k}|) + \omega(k), \tag{4}$$

where $\omega(k)$ is the phonon frequency, and k is its wave vector.* At small k, $\omega(k)$ becomes

$$\omega(k) = ck - \alpha k^2 \tag{5}$$

(c is the velocity of sound). We assume that $\alpha > 0$, i.e., that the phonon Spectrum is stable (see footnote in preceding column). The function $\epsilon(p)$ has a singularity at $p = p_c$. We assume, however (this will be con-

*We use a system of units in which $h = 1$.

firmed later on), that this singularity appears only in terms of higher order of smallness than second, i.e., that

$$\varepsilon(p_c + \Delta p) \approx \varepsilon_c + c\Delta p + \beta(\Delta p)^2. \quad (6)$$

(We have assumed the speed of the excitation to be c at $p = p_c$.)

When $p = p_c$ and $\cos\vartheta = 1$ (ϑ is the angle between k and p) the right half of (4) becomes, taking (5) and (6) into account,

$$\varepsilon_c + (\beta - \alpha)k^2. \quad (7)$$

The point $p = p_c$ is actually the threshold only if (7) has a minimum at $k = 0$, i.e., if the inequality $\beta > \alpha$ is satisfied.

Since in this case the excitation can produce a phonon with $k \to 0$ when $p = p_c$, the region of small values of the argument of one of the Green's functions, say q, will be significant in the integral of (3). If $\omega \gg \epsilon_c$ and $q = k \ll p_c$, the Green's function $G(q)$ is the propagation function of the phonon and has the form

$$G^{-1}(k) = B^{-1}[\omega^2(k) - \omega^2 - i\delta]$$
$$\approx B^{-1}[(ck - \alpha k^2)^2 - \omega^2 - i\delta], \quad \delta \to +0. \quad (8)$$

When $p \approx p_c$ and $\epsilon \approx \epsilon_c$, the function $G^{-1}(p)$ has a singularity. We assume, however, that according to (6) $G^{-1}(p)$ has the following form near zero (i.e., near the pole of G)

$$G^{-1}(p) = A^{-1}[c\Delta p - \Delta\varepsilon + \beta(\Delta p)^2 - i\delta]$$
$$(\Delta p = p - p_c, \ \Delta\varepsilon = \varepsilon - \varepsilon_c) \quad (9)$$

plus certain terms of higher order, which we must determine.

At small k, the vertex portion Γ_0, as is known, has the form $g_0 k$. We assume that when $k \to 0$, Γ is also proportional to k:

$$\Gamma(p, k, p - k) = gk \quad \text{for } k \to 0. \quad (10)$$

In this case the integral equation (3) for G can be solved by successive approximation.

We separate the region of integration in (3) into two regions — one of large k and ω ($k \geq K$ and $\omega \geq \Omega$) and one of small values of k and ω ($k \leq K$ and $\omega \leq \Omega$), with

$$\Delta p \ll K \ll p_c, \quad \Delta\varepsilon \ll \Omega \ll \varepsilon_c. \quad (11)$$

In the second region it is possible to use expressions (8) through (10) for $G(k)$, $G(p-k)$ and Γ. Assuming that the integral over large k and ω has no singularities, we obtain from (3)

$$G^{-1} \backsim i\int \frac{k^4 dk d(\cos\vartheta) d\omega}{[(ck - \alpha k^2)^2 - \omega^2 - i\delta][c(\Delta p - k\cos\vartheta) - \Delta\varepsilon + \omega + \beta(\Delta p - k)^2 - i\delta]}. \quad (12)$$

The symbol $\backsim$ here and hereinafter denotes that the right half of the equality may differ from the left half by a coefficient that has no singularities and by an additive term. We shall henceforth drop the regular coefficients, without stating it specifically.

Small angles $\vartheta \ll 1$ are significant in the integral (12), and we can therefore put, with sufficient accuracy, $\cos\vartheta = 1$ in the quadratic terms. The integration over ω can be extended to the interval $-\infty < \omega < +\infty$. As a result, the integration reduces to taking the residue with respect to ω at $\omega = ck - \alpha k^2$. We have

$$G^{-1}(p) \backsim \int \frac{k^3 dk d(\cos\vartheta)}{x + ck(1 - \cos\vartheta) - 2\beta\Delta pk - k^2(\alpha - \beta)}$$
$$\backsim \int k^2 \ln[x - 2\beta k\Delta p + (\beta - \alpha)k^2]\, dk. \quad (13)$$

We introduce here $x = c\Delta p + \beta(\Delta p)^2 - \Delta\epsilon$. Factoring the expression under the logarithm sign and integrating, we get

$$G^{-1}(p) \backsim a_1(k_1/2)^3 \ln k_1 + a_2(k_2/2)^3 \ln k_2, \quad (14)$$

where

$$k_{1,2} = \beta\Delta p \pm \sqrt{(\beta\Delta p)^2 - (\beta - \alpha)x}.$$

It is seen from (14) that G^{-1} really has a singularity in terms of higher order than those used in Eq. (9), thus confirming the assumption made.

We determine $G^{-1}(p)$ in the direct vicinity of the pole of $G(p)$, namely at

$$|x| \ll \beta(\Delta p)^2. \quad (15)$$

In this case the term with k_2 can be neglected. We then obtain

$$G^{-1} \backsim (\Delta p)^3 \ln(-\Delta p) \quad (16)$$

or, taking (9) into account,

$$G^{-1}(p) = A^{-1}[c\Delta p + \beta(\Delta p)^2$$
$$+ a(\Delta p)^3 \ln(-\Delta p) - \Delta\varepsilon]. \quad (17)$$

Equation (17) determines the energy of the elementary excitation near the threshold:

$$\varepsilon(p) = \varepsilon_c + c(p - p_c)$$
$$+ \beta(p - p_c)^2 + a(p - p_c)^3 \ln(p_c - p). \quad (18)$$

Thus, when $p > p_c$, the excitation energy has a negative imaginary part equal to $-a\pi(\Delta p)^3$. This means that when $p > p_c$ the excitations are damped out, and their lifetime is inversely proportional to $(p - p_c)^3$. We note that the same result would have been obtained by perturbation theory, since the interaction with long-wave phonons is always weak, owing to the presence of the factor k in Γ.

Knowing the Green's function it is easy to verify that all the corrections to Γ have an order not less than k, thereby justifying the assumption that Γ is proportional to k.

b. Properties of the spectrum near the threshold of disintegration into two excitations with parallel non-zero momenta. In this case, when integrating over q in (3), the significant values of the 4-momentum q, as expected from physical considerations, are those with which the excitations are created near threshold. However, the values of the momentum and energy at which the excitations are created near threshold are not singular for the Green's functions. The only singularity of this point is that the given excitations may "coalesce" with the other in the vicinity of this point, a process which is impossible at absolute zero, owing to the absence of real excitations. Therefore the Green's functions under the integral in (3) have near the pole the usual form

$$G^{-1}(q) = A^{-1}[\varepsilon(q) - \omega - i\delta], \quad \delta \to +0. \tag{19}$$

This circumstance makes it much easier to investigate the problem.

Let us examine the right half of (3). The quantity Γ_0 in this equation has naturally no singularity at $p = p_c$. We also assume that the total vertex portion neither vanishes nor goes to infinity at the threshold.

Let us break up, as in the case of phonon production, the region of integration over q in Eq. (3) into a small one, near the values of momentum and energy q_0 and ϵ_0 with which the excitations are created*

$$|\mathbf{q} - \mathbf{q}_0| \leqslant K \ll p_c, \quad |\omega - \varepsilon_0| \leqslant \Omega \leqslant \varepsilon_c$$

and a large one. In the small region Γ and Γ_0 can be assumed constant. As a result

$$G^{-1}(p) \backsim i \int \frac{d\omega d^3\mathbf{q}}{[\varepsilon(q) - \omega - i\delta][\varepsilon(|\mathbf{p} - \mathbf{q}|) - \varepsilon + \omega - i\delta]} \backsim \int \frac{d^3\mathbf{q}}{\varepsilon(q) + \varepsilon(|\mathbf{p} - \mathbf{q}|) - \varepsilon}. \tag{20}$$

Since the expression $\epsilon(q) + \epsilon(|\mathbf{p} - \mathbf{q}|)$ should have a minimum at $p = p_c$, it has the following form at values of p close to p_c

$$\varepsilon(q) + \varepsilon(|\mathbf{p} - \mathbf{q}|) \approx \varepsilon_c + v_c \Delta p + \alpha(\mathbf{q} - \mathbf{q}_0)^2 + \beta(\mathbf{q} - \mathbf{q}_0, \mathbf{p}_c)^2/p_c^2, \tag{21}$$

where v_c is the velocity of each of the excitations produced at the threshold point, $\mathbf{q}_0 = \nu \mathbf{p}_c$ is the momentum of one of the created excitations, and α and β are coefficients that are determined by the type of the function $\epsilon(q)$ and $\epsilon(|\mathbf{q} - \mathbf{p}|)$:

$$\alpha = \frac{v_c p_c}{2q_0(p_c - q_0)},$$

$$\beta = \frac{1}{2}\left\{\left(\frac{\partial^2\varepsilon}{\partial q^2}\right)_{q=q_0} + \left(\frac{\partial^2\varepsilon}{\partial q^2}\right)_{q=p_c-q_0} - \frac{v_c p_c}{q_0(p_c - q_0)}\right\}.$$

Introducing a new variable $\mathbf{u} = \mathbf{q} - \mathbf{q}_0$, $\mathbf{u}\mathbf{p}_c = up_c \cos\psi$, we get

$$G^{-1} \backsim \int \frac{u^2 du d(\cos\psi)}{v_c\Delta p - \Delta\varepsilon + \alpha u^2 + \beta u^2 \cos^2\psi} \backsim \sqrt{v_c\Delta p - \Delta\varepsilon}. \tag{22}$$

The square root in (22) should be taken with the plus sign, since when $v_c\Delta p - \Delta\epsilon > 0$, the integral in (22) is positive. Since the point $p = p_c$, $\epsilon = \epsilon_c$ is by definition a point of the spectrum, $G^{-1}(p)$ should vanish at $\Delta p = 0$ and at $\Delta\epsilon = 0$, and consequently at small values of $\Delta\epsilon$ and Δp the regular portion of $G^{-1}(p)$ should have the form $a'\Delta p + b'\Delta\epsilon$. Finally

$$G^{-1}(p) = A_1^{-1}[a\Delta p + \Delta\varepsilon + b\sqrt{v_c\Delta p - \Delta\varepsilon}]. \tag{23}$$

The excitation energy is determined by the equation

$$G^{-1}(p) = 0. \tag{24}$$

For this equation to have a solution at $p < p_c$, it is necessary to satisfy the inequality

$$(a + v_c)/b > 0. \tag{25}$$

Now the solution of the Eq. (24) is of the form

$$\varepsilon = \varepsilon_c + v_c(p - p_c) - \left(\frac{a + v_c}{b}\right)^2 (p - p_c)^2. \tag{26}$$

When $p > p_c$, Eq. (24) has no solutions at all for ϵ in the vicinity of ϵ_c, neither real nor complex. Thus, in this case the curve of the energy spectrum approaches the threshold point with a slope equal to v_c, and does not continue further. It is easy to verify then that all the corrections to Γ are finite, thus justifying the assumption that Γ is finite at the threshold point.

c. Disintegration into two excitations emitted at an angle to each other. In this case, as in the preceding one, the region of importance in the integration is of those values of q, with which the excitations are produced near the threshold point. In this region the Green's functions have the usual form (19). However, one cannot state now that the vertex Γ is finite when $\epsilon = \epsilon_c$. To clarify the character of the singularity $\Gamma(p; q, p - q)$ we express it in terms of $\Gamma_0(p; q, p - q)$ and an irreducible 4-particle vertex part $\gamma_1(q_1, p - q; q, p - q)$, i.e., the set of all the 4-particle diagrams, which cannot be divided between the end points $q_1, p - q_1$ and $q, p - q$ into two parts, connected only by one or two lines. In order to ex-

*An important region in the integral of (3) is also the symmetrical region $q \approx p_c - q_0$, $\omega = \varepsilon_c - \varepsilon_0$. This region makes exactly the same contribution as the first one, and we shall not write down the corresponding terms.

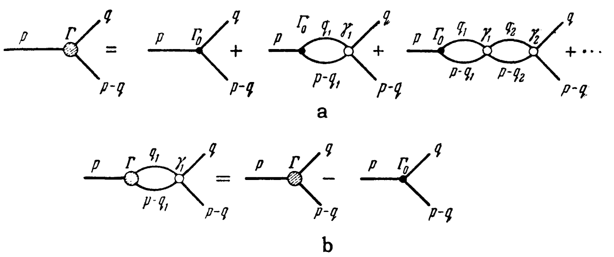

FIG. 2

press Γ in terms of Γ_0 and γ_1, it is necessary either to sum the series presented in Fig. 2a, or to solve the integral equation, shown in Fig. 2b, which is a result of this series. Theoretically this expression can be written in the following form*

$$\Gamma(p;q,p-q)-\Gamma_0(p;q,p-q)$$
$$=i\int\Gamma(p,q_1,p-q_1)\,G(q_1)\,G(p-q_1)$$
$$\times\gamma_1(q_1,p-q_1;p,p-q)\,d^4q_1/(2\pi)^4. \qquad (27)$$

We emphasize that this is an exact equation.

Since the quantity γ_1 does not contain at all a singular integration of the type contained in Eq. (3) it is natural to assume that, like Γ_0, it remains finite at the threshold point.

We also assume that $\Gamma(p,q,p-q)$ has near the threshold a singularity only with respect to the first argument. (Each term of the series of Fig. 2a has this property.) All these assumptions should be verified in the future.

Separating in (27), as customary when integrating over the frequency ω_1, the region $|\omega_1-\dot{\Delta}|\ll\Delta$ and carrying out the integration, we obtain

$$\Gamma(p;q,p-q)-\Gamma_0(p;q,p-q)$$
$$\backsim\int\frac{\Gamma(p;q_1,p-q_1)\,\gamma_1(q_1,p-q_1;q,p-q)}{\varepsilon(q_1)+\varepsilon(|\mathbf{p}-\mathbf{q}_1|)-\varepsilon}\,d^3\mathbf{q}_1. \qquad (28)$$

When integrating over $\mathbf{q}_1$ we separate the following region: $|\mathbf{q}_1-\mathbf{p}_0|\ll p_0$, $||\mathbf{q}_1-\mathbf{p}|-p_0|\ll p_0$. In this region Γ and γ_1 can be considered independent of $\mathbf{q}_1$. Considering that in this region $\epsilon(q_1)$ and $\epsilon(|\mathbf{p}-\mathbf{q}_1|)$ have the form given by Eq. (2), we get

$$\Gamma(p)-\Gamma_0(p)\backsim\Gamma(p)$$
$$\times\int\frac{d^3\mathbf{q}_1}{2\Delta-\varepsilon+(q_1-p_0)^2/2\mu+(|\mathbf{q}_1-\mathbf{p}|-p_0)^2/2\mu} \qquad (29)$$

*With the aid of Eq. (27) it is easy to determine $\Gamma(p;q,p-q)$ for case b. It turns out that in this case $\Gamma(p_c+\Delta p;q,p_c+\Delta p-q)$ is of the form

$$\Gamma\approx P+Q\sqrt{v_c\Delta p-\Delta\varepsilon}.$$

As is to be expected, this expression approaches the constant limit P when Δp and $\Delta\varepsilon$ go to zero.

(We recall that the symbol $\backsim$ denotes equality with accuracy to a regular coefficient and a regular added term.)

Let us change to cylindrical coordinates q_z', q_ρ', and φ, in accordance with the following formulas (the z axis is aligned with p):

$$q_z=p_0\cos\vartheta_1+q_z',\quad q_x=(p_0\sin\vartheta_1+q_\rho')\cos\varphi,$$
$$q_y=(p_0\sin\vartheta_1+q_\rho')\sin\varphi, \qquad (30)$$

where the angle ϑ_1 is given by

$$2p_0\cos\vartheta_1=p. \qquad (31)$$

Inserting (30) into (29) and neglecting the higher powers of q_z' and q_ρ', we obtain

$$\Gamma(p)-\Gamma_0(p)\backsim\Gamma(p)\int\frac{dq_\rho'\,dq_z'}{2\Delta-\varepsilon+(\sin^2\vartheta_1q_\rho'^2+\cos^2\vartheta_1q_z'^2)/\mu}$$

or, introducing the polar coordinates r and ψ,

$$\sin\vartheta_1q_\rho'/\sqrt{\mu}=r\cos\psi,\quad\cos\vartheta_1q_z'/\sqrt{\mu}=r\sin\psi,$$
$$\Gamma(p)-\Gamma_0(p)\backsim\Gamma(p)\int\frac{r\,dr}{2\Delta-\varepsilon+r^2}\backsim\Gamma(p)\ln(2\Delta-\varepsilon). \qquad (32)$$

From (32) we find $\Gamma(p)$ at $|2\Delta-\epsilon|\ll\Delta$:

$$\Gamma(p;q,p-q)=P\left[1+Q\ln\frac{2\Delta-\varepsilon}{\alpha}\right]^{-1}$$
$$\approx\frac{P}{Q}\left[\ln\frac{2\Delta-\varepsilon}{\alpha}\right]^{-1}-\frac{P}{Q^2}\left[\ln\frac{2\Delta-\varepsilon}{\alpha}\right]^{-2}, \qquad (33)$$

where P and Q are functions that have no singularities at $\epsilon=2\Delta$. The expression (33) for Γ should be inserted into Eq. (3) to determine $G(p)$. The integration in (3) is fully analogous to the integration in (27) and yields a term proportional to $\ln\frac{2\Delta-\epsilon}{\alpha}$. Thus

$$G^{-1}\sim[\ln(2\Delta-\varepsilon)+B']\left[1+Q\ln\frac{2\Delta-\varepsilon}{\alpha}\right]^{-1}$$
$$\backsim\left(\ln\frac{2\Delta-\varepsilon}{\alpha}\right)^{-1}. \qquad (34)$$

Finally, considering that by definition $G^{-1}(p_c)=0$, we find

$$G^{-1}(p)=A_1^{-1}\left[(p-p_c)-a\left(\ln\frac{2\Delta-\varepsilon}{\alpha}\right)^{-1}\right]. \qquad (35)$$

Thus, in this case the spectrum at $p<p_c$ has the form

$$\varepsilon(p)=2\Delta-\alpha e^{-a/(p-p_c)}. \qquad (36)$$

In this case, too, the curve $\epsilon(p)$ terminates at the point $p=p_c$ and has at this point a horizontal tangent of infinite order.

Knowing the Green's function, it is easy to check the assumption, made in the solution of Eq. (27), that the vertex part γ_1 is finite at the threshold point.

We note that in all the foregoing cases the

Green's function has a branch point at $\epsilon = \epsilon_c$, $p = p_c$.

The experimental data available at the present time do not permit an unambiguous answer to the question of how the phonon spectrum in He^4 is terminated. There probably occurs here either an emission of a phonon with zero momentum (case a) or a disintegration into two rotons with $\epsilon = \Delta$ (case c).

3. INELASTIC SCATTERING OF NEUTRONS WITH CREATION OF EXCITATIONS NEAR THE DISINTEGRATION THRESHOLD

The most effective method of investigating the excitation spectra is inelastic scattering of neutrons. In connection with this, we shall consider briefly the question of the probability of inelastic neutron scattering with a neutron energy loss $\epsilon \approx \epsilon_c$ and with a momentum loss $p \approx p_c$, i.e., with creation of excitations of energies close to their disintegration threshold.

In inelastic neutron scattering, the following energy and momentum conservations should be satisfied

$$P_1^2/2m = P_2^2/2m + \varepsilon, \tag{37}$$

$$\mathbf{p} = \mathbf{P}_1 - \mathbf{P}_2, \tag{38}$$

where m is the neutron mass, ϵ and $\mathbf{p}$ are the energy and momentum of the excitations created by the neutron, while $\mathbf{P}_1$ and $\mathbf{P}_2$ are respectively the initial and final momenta of the neutron. Squaring (38) we get

$$p^2 = 2m\left(E_1 + E_2 - 2\sqrt{E_1 E_2}\cos\varphi\right), \tag{39}$$

where E_1 and E_2 are neutron energies before and after scattering, and φ is the scattering angle. When $p \approx p_c$ and $\epsilon \approx \epsilon_c$, the neutrons are scattered through angles $\varphi \approx \varphi_c$, where φ_c is determined by

$$p_c^2 = 2m\left(2E_1 - \varepsilon_c - 2\sqrt{E_1(E_1 - \varepsilon_c)}\cos\varphi_c\right). \tag{40}$$

The energy and momentum transfer in the scattering angle are connected here by an equation that follows from (39) and (40):

$$p - p_c = \frac{m}{p_c}\left[\left(\sqrt{\frac{E_1}{E_1 - \varepsilon_c}}\cos\varphi_c - 1\right)(E_1 - E_2 - \varepsilon_c) + \sqrt{E_1(E_1 - \varepsilon_c)}\sin\varphi_c(\varphi - \varphi_c)\right]$$
$$= \beta\varphi' + \nu\varepsilon' \quad (\varphi' = \varphi - \varphi_c, \quad \varepsilon' = \varepsilon - \varepsilon_c). \tag{41}$$

If the momentum transfer p is close to the threshold of photon creation, then the distribution of the scattered neutrons has a sharp line when $p < p_c$ and as $p \to p_c$ the intensity of this line has no singularity whatever. When $p > p_c$ the line starts broadening in proportion to $(p - p_c)^3$. The distribution of the scattered neutrons has in this case the form

$$dw = c\frac{a\pi(\beta\varphi' + \nu\varepsilon')^3}{[\varepsilon'(1 - \nu c) - \beta c\varphi']^2 + a^2\pi^2(\beta\varphi' + \nu\varepsilon')^6}\,d\varphi' d\varepsilon'. \tag{42}$$

We now proceed to scattering with creation of excitations near the threshold of disintegration into two excitations with finite momenta. To be specific, we shall treat the third type of disintegration. In this case, one excitation will be created at $\epsilon > 2\Delta$ and two excitations, each with energy $\geq \Delta$, will be created at $\epsilon > 2\Delta$. Graphically these processes are presented in Fig. 3. The wavy line corresponds here to the free neutron, the point $V(P, P_2; q, P_1 - P_2 - q)$ denotes the amplitude of the scattering of the neutron by the free atom, and $\gamma_2(q_1, p - q_1; q, p - q)$ denotes the aggregate of all 4-particle vertex diagrams, which cannot be separated between the terminals $q_1, p - q_1$ and $q, p - q$ into two parts joined by a single line only. We have taken into account here that the interaction between the free neutron and the atom can always be considered weak, and therefore we neglected all the graphs where the vertex V is encountered two times or more.

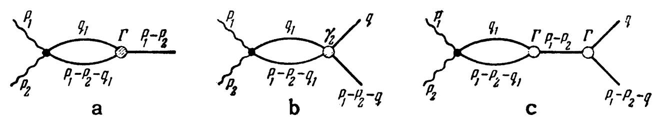

FIG. 3

When $\epsilon < 2\Delta$, when only one excitation can be created, ϵ and p are related by Eq. (36), which together with Eq. (41), in which we put $\epsilon_c = 2\Delta$, determines the energy lost by the neutron as a function of the scattering angle

$$E_1 - E_2 - 2\Delta = \alpha e^{-a'/(\varphi_c - \varphi)},$$
$$a' = ap_c/m\sqrt{E_1(E_1 - 2\Delta)}\sin\varphi_c. \tag{43}$$

The probability of neutron scattering in the momentum integral is given by the expression

$$dw = 2\pi|N_1 M|^2\,\delta(E_2 + \varepsilon - E_1)\,d^3\mathbf{P}_2/(2\pi)^3. \tag{44}$$

Here $M(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_1 - \mathbf{P}_2)$ is the matrix element corresponding, in accordance with the usual rule, to the graph of Fig. 3a, and N_1 is the renormalization constant, equal to the square root of the negative residue of the Green's function of the created excitation at the point corresponding to the excitation energy. (Such a renormalization factor must be written down for each free end of the graph.) From (35) we find

$$N_1 = \sqrt{\frac{A(2\Delta-\varepsilon)}{a}}\left[-\ln\frac{2\Delta-\varepsilon}{\alpha}\right]. \qquad (45)$$

As regards the matrix element M, it approaches, when ϵ approaches 2Δ, a constant limit, since the logarithm arising as a result of integration with respect to q (see Fig. 3a) is cancelled by the logarithm from the numerator in Γ (V, naturally, has no singularity). Therefore the behavior of dw is determined by the behavior of N_1, and the angular distribution of the neutrons (when $\varphi \le \varphi_0$) has the form

$$dw = C_1(\varphi_c - \varphi)^{-2} e^{-a'/(\varphi_c-\varphi)}\, d\varphi. \qquad (46)$$

Thus, the probability of creation of excitation as $\varphi \to \varphi_c$ tends rapidly to zero.

We now proceed to scattering of neutrons with transfer of an energy greater than 2Δ. The scattering probability in this case has the form

$$dw = 2\pi\,|N_2^2 M_2|^2\, \delta(E_2+\varepsilon-E_1)\frac{d^3\mathbf{P}_2}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}. \qquad (47)$$

Here M_2 is the sum of the matrix elements corresponding to the graphs of Figs. 3b and 3c. Now the normalization factors N_2, pertaining to excitations with energies close to Δ, have no singularities. Nor does the matrix element of the diagram in Fig. 3b have a singularity, since the vertex γ_2, like Γ, has a factor $[\ln(2\Delta-\epsilon)]^{-1}$, which, as can be readily seen, cancels out the integral with respect to q_1. The singularity of M_2 is therefore given by the graph of Fig. 3c. The matrix element of this graph has a maximum at $\varphi < \varphi_c$ and gives in this region the main contribution to the scattering. The corresponding probability has the form

$$dw = 2\pi\left|\int V(P_1, P_2, q_1, P_1-P_2-q_1)\right.$$
$$\left.\times G(q_1)\,G(P_1-P_2-q_1)\,\Gamma(q_1, P_1-P_2-q_1, P_1-P_2)\frac{d^4q_1}{(2\pi)^4}\right|^2$$
$$\times N_2^4|\Gamma(P_1-P_2; q, P_1-P_2-q)\,G(P_1-P_2)|^2$$
$$\times\delta(E_2+\varepsilon-E_1)\frac{d^3\mathbf{P}_2}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}. \qquad (48)$$

The singularity of M_2 is contained in the term with the product ΓG. Integrating over q and using (33) and (35) we get

$$dw = C_2\, d\varphi\, dE_2\left\{\left[a-(p-p_c)\ln\left|\frac{2\Delta-E_1+E_2}{\alpha}\right|\right]^2 + \pi^2(p-q_c)^2\right\}^{-1}. \qquad (49)$$

Inserting (41) into (49) we obtain the distribution of the scattered neutrons by angles and by energies

$$dw = C_2\, d\varepsilon'\, d\varphi'\left\{\left[a-\beta\varphi'\ln\frac{\varepsilon'}{\alpha}\right]^2+\pi^2\varphi'^2\beta^2\right\}^{-1}. \qquad (50)$$

Let us describe qualitatively the scattering of the neutrons in this case. When $\varphi < \varphi_c$ there is a sharp line, corresponding to the creation of one roton, and its intensity tends to zero when $\varphi \to \varphi_c$. The creation of two rotons leads to a continuous spectrum of scattered neutrons, and the scattering intensity has a minimum at $\epsilon = 2\Delta$; when $\varphi < \varphi_c$ it has a smeared maximum at

$$\varepsilon' = \alpha e^{-a'/(\varphi_c-\varphi)}.$$

The calculations for the case of disintegration into two excitations with parallel momentum (case b) are analogous. We shall cite only the final formulas. The scattering probability with energy loss $\epsilon < \epsilon_c$ is

$$dw = C_1\sqrt{\varphi_c-\varphi}\, d\varphi. \qquad (51)$$

The probability of scattering with $\epsilon' = \epsilon - \epsilon_c > 0$ is of the form

$$dw_1 = C_2\, d\varepsilon'\, d\varphi'\,[(a+v_c)\varepsilon' + b\sqrt{(v_c\nu-1)\varepsilon'-\beta\varphi'}]^{-2} \text{ for } (v_c\nu-1)\varepsilon'-\beta\varphi' > 0,$$
$$dw = C_2\, d\varepsilon'\, d\varphi'\{(a+v_c)^2\varepsilon'^2 + b^2[(v_c\nu-1)\varepsilon'-\beta\varphi']\}^{-1} \text{ for } (v_c\nu-1)\varepsilon'-\beta\varphi' < 0. \qquad (52)$$

In this work we did not investigate the case when the threshold of disintegration into three excitations lies below the threshold of disintegration into two excitations. Although logically such a possibility does exist, it is of little likelihood.

Our entire investigations pertain naturally not only to the phonon spectrum of liquid He^4, but to any Bose excitation mode in condensed bodies. If, however, the interaction that leads to the excitation disintegration is weak, all the singularities described will appear only within a very small momentum interval near p_c, making it possible to determine them experimentally. It must also be borne in mind that in cases a and b, at weak interactions, the curve of the spectrum may, after a brief interaction, continue again with damping. Furthermore in the case of crystals the situation becomes greatly complicated by anisotropy.

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