THE MASS SPECTRUM OF MESONS IN HEISENBERG'S THEORY

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The mass spectrum of mesons is computed in Heisenberg's theory. The comparison of the results with the experimental data indicates that the scalar version of the nonlinear term leads to the best agreement in the first approximation of the Tamm-Dancoff method.

THE theory of elementary particles proposed by Heisenberg[1] makes use of the Lagrangian

$$L = \bar{\psi}\gamma_\mu \nabla_\mu \psi - {}^1/_2\, l^2 (\bar{\psi}\psi)(\bar{\psi}\psi) \tag{1}$$

and the new commutator function

$$\frac{1}{2} S(q) = \frac{\varkappa^3}{q^2(q^2+\varkappa^2)}\left(\frac{\varkappa}{q^2}\gamma_\nu q_\nu - i\right). \tag{2}$$

The mass spectrum of the mesons, and the mass and electric charge of the nucleons were computed on this basis;[2,3] the results are quite close to what is observed in nature. It becomes necessary, in this connection, to investigate other versions of the theory in order to determine whether the simplest choice of the nonlinear term, as made in reference 1, is also the best.

The nonlinear term in (1) can have the form

$$\pm {}^1/_2\, l^2 (\bar{\psi} O_n \psi)(\bar{\psi} O_n \psi), \tag{3}$$

where $O_n = 1,\ \gamma_5,\ \gamma_\mu,\ i\gamma_5\gamma_\mu,\ i\gamma_\mu\gamma_\nu\ (\mu \neq \nu)$ are matrices known from the theory of β decay. The form of the matrices O_n and the sign of (3) have to be chosen such that the consequences of the theory be in agreement with experiment.

1. MESONS IN THE MODEL WITH THE NONLINEAR TERM (3)

We choose the minus sign in front of the nonlinear term of the Lagrangian.

The integral form of the field equation,

$$\psi(x) = -\frac{1}{2} i l^2 \int G(x, u) O_n \psi(u) \cdot \bar{\psi}(u) O_n \psi(u)\, d^4u, \tag{4}$$

applied to the meson amplitude

$$\varphi(x, y) = \langle 0 | T\psi(x)\bar{\psi}(y) | \Phi \rangle, \tag{5}$$

leads to the equation

$$\varphi(p) = \frac{il^2}{8(2\pi)^4}\int d^4q [\, G(p+q) O_n \varphi(p) O_n S(q) - G(p+q) O_n S(q)\, \mathrm{Sp}\, O_n \varphi(p) - S(-q) O_n \varphi(p) O_n G(p+q) + S(-q) O_n G(p+q)\, \mathrm{Sp}\, O_n \varphi(p)], \tag{6}$$

where

$$\varphi(p) = \int \langle 0 | T\psi(x)\bar{\psi}(x) | \Phi \rangle e^{-ipx} d^4x. \tag{7}$$

Equation (6) is only approximate, since we neglected the amplitude $\langle 0 | N\psi(x)\psi(u)\bar{\psi}(u)\psi(u) | \Phi \rangle$ in its derivation, which corresponds to the first approximation in the Tamm-Dancoff method.

Substituting expression (2) and the Green's function

$$G(p+q) = 2(p+q)^{-2}\gamma_\mu (p_\mu + q_\mu), \tag{8}$$

into (6), we obtain

$$\varphi = \gamma_\mu O_n \varphi O_n \gamma_\nu J_{\mu\nu} + {}^1/_2 (\gamma_\mu O_n \varphi O_n - O_n \varphi O_n \gamma_\mu) J_\mu - \gamma_\mu O_n \gamma_\nu J_{\mu\nu}\, \mathrm{Sp}\, \varphi O_n - {}^1/_2 (\gamma_\mu O_n - O_n \gamma_\mu) J_\mu\, \mathrm{Sp}\, \varphi O_n, \tag{9}$$

where J_μ and $J_{\mu\nu}$ are momentum integrals:

$$J_\mu = \frac{l^2\varkappa^3}{(2\pi)^4}\int \frac{(p_\mu + q_\mu)\, d^4q}{q^2(q^2+\varkappa^2)(p+q)^2} = p_\mu B(p^2),$$

$$J_{\mu\nu} = \frac{il^2\varkappa^4}{(2\pi)^4}\int \frac{(p_\mu + q_\mu)\, q_\nu d^4q}{q^4(q^2+\varkappa^2)(p+q)^2} = \delta_{\mu\nu} C(p^2) + p_\mu p_\nu D(p^2), \tag{10}$$

which are expressed in terms of the functions B, C, and D introduced in reference 2.

In the center of mass system of the meson equation (9) simplifies if φ is expanded in terms of the 16 Dirac matrices:

$$\varphi = \frac{1}{4}\sum_{\rho=1}^{16} \varphi_\rho \Gamma_\rho, \tag{11}$$

since we can then use the following formulas:

$$O_n \Gamma_\rho O_n = A_\rho^{(n)} \Gamma_\rho, \quad \gamma_\mu \Gamma_\rho \gamma_\mu = \beta_\rho \Gamma_\rho, \quad \gamma_4 \Gamma_\rho \gamma_4 = \varepsilon_\rho \Gamma_\rho \tag{12}$$

(A_ρ, β_ρ, and ϵ_ρ are numbers; no sum over ρ). Multiplying (9) by Γ_ρ and taking the spur, we obtain the equation

$$\varphi_\rho = \varphi_\rho A_\rho (\beta_\rho C + \varepsilon_\rho p^2 D) + \varphi_\sigma A_\sigma \frac{1-\varepsilon_\sigma}{2} p_4 B - 4\left[\delta_{\rho n}(\beta_n C + \varepsilon_n p^2 D) + \delta_{\sigma n}\frac{1-\varepsilon_n}{2} p_4 B\right]\varphi_n, \tag{13}$$

where $\varphi_\rho = \mathrm{Sp}\, \varphi\Gamma_\rho$, $\varphi_\sigma = \mathrm{Sp}\, \varphi\Gamma_\sigma$, and $\Gamma_\sigma = \gamma_4\Gamma_\rho$.

This equation together with the equation for φ_σ forms a system of two homogeneous equations. By setting its determinant equal to zero, we obtain the equation

$$(a_\rho Dp^2 + b_\rho C - 1)(a_\sigma Dp^2 + b_\sigma C - 1) + eB^2p^2 = 0, \quad (14)$$

which can be used to determine p^2. Its coefficients are given by the formulas

$$a_\rho = \varepsilon_\rho (A_\rho - 4\delta_{\rho n}), \qquad b_\rho = \beta_\rho (A_\rho - 4\delta_{\rho n}),$$
$$e = -\tfrac{1}{2}(1 - \varepsilon_\rho)(A_\rho - 4\delta_{\rho n})(A_\sigma - 4\delta_{\sigma n}) \quad (15)$$

and depend on the choice of the nonlinear term as well as on the spin and the parity of the emitted meson. All of them are easily computed with the help of (12).

The functions B, C, and D contain the constant $(4\pi/\kappa l)^2$ as a factor. We calculated them, following reference 2, by determining the mass of the proton and then identifying it with κ. Thereafter the functions B, C, and D were tabulated and Eq. (14) solved numerically, The results of the calculation and the values of the constant are given in Table I.

It should be noted, first of all, that our results for the scalar version are somewhat different from the data of Heisenberg, Kortel, and Mitter.[2] The reason for this is that these authors include only the first two terms of our equation (6), so that the coefficient e in (14) does not contain the factor $(1 - \epsilon_\rho)/2$. However, the term eB^2p^2 is relatively small, and the discrepancy does not exceed ~ 7 to 10%.

In the vector version the mesons are absent altogether. By changing the sign in front of the nonlinear term, i.e., going over to the Lagrangian

$$L' = \bar{\psi}\gamma_\mu \nabla_\mu \psi + \tfrac{1}{2} l^2 (\bar{\psi} O_n \psi)(\bar{\psi} O_n \psi), \quad (16)$$

we can change the sign of the interaction potential between the nucleon and the antinucleon, which also modifies the mass spectra of the mesons. The corresponding equations are obtained from those above by makeing the replacement $l^2 \to -l^2$. The results of the solution of these altered equations are given in the last column of Table I.

It is seen from these data that the masses of the pseudoscalar mesons, as a rule, exceed the observed masses by a factor of 2 to 3. It should be expected that the results of the theory could be improved by taking account of the isotopic properties of the field ψ.

2. MESONS IN THE "REALISTIC MODEL"

We introduce the isotopic properties of the field ψ in the framework of the "realistic model" of Heisenberg,[1] which makes use of the iso-scalar χ together with the iso-spinor ψ in defining the Lagrangian

$$L = \bar{\psi}\gamma_\mu \nabla_\mu \psi + \bar{\chi}\gamma_\mu \nabla_\mu \chi - l^2 (\bar{\psi} O_n \chi)(\bar{\chi} O_n \psi). \quad (17)$$

In the first approximation of the Tamm-Dancoff

TABLE I

Version	$\left(\frac{4\pi}{\kappa l}\right)^2$	Spin and parity	M/M_p for L	M/M_p for L′
S	2.897	0+	{0.40, 0.12}	—
		0−	0.23	0.23
		1+	0.05	0.05
		1−	—	0.05
V	5.078	0+	—	0.45
		0−	—	—
		1+	—	0.30
		1−	—	0.07
T	4.738	0+	—	0.56
		0−	0.56	—
		1+	—	—
		1−	—	—
A	7.645	0+	0.06	0 35
		0−	0.68	0.68
		1+	<0.03	—
		1−	0.20	—
P	1.665	0+	—	{0.386, 0.392}
		0−	{0.55, 0.71}	0.55
		1+	0.13	—
		1−	0.16	—

method the meson amplitude $\varphi(x, y)$ satisfies the equation

$$\varphi(x, y) = \frac{il^2}{8}\int d^4u\,[G(x, u)\,O_n\chi(u, u)\,O_n S^{\psi}(u, y) + S^{\psi}(x, u)\,O_n\chi(u, u)\,O_n G(u, y)], \qquad (18)$$

where

$$\chi(u, v) = \langle 0\,|\,T\chi(u)\overline{\chi}(v)\,|\,\Phi\rangle. \qquad (19)$$

The equation for $\chi(u, v)$ is of an analogous form:

$$\chi(x, y) = \frac{il^2}{8}\int d^4u\,[G(x, u)\,O_n\,\mathrm{tr}\varphi(u, u)\,O_n S^{\chi}(u, y) + S^{\chi}(x, u)\,O_n\,\mathrm{tr}\,\varphi(u, u)\,O_n G(u, y)] \qquad (20)$$

(tr denotes the spur with respect to the isotopic spin indices).

We retain the form (2) for the function S^{χ}, while S^{ψ} is taken to be[1]

$$\frac{1}{2}S^{\psi}(q) = \frac{\varkappa^3}{q^2(q^2+\varkappa^2)}\left[\frac{\varkappa}{q^2}\gamma_\nu q_\nu\tau_0 - i\tau_3\right], \qquad (21)$$

where τ_0 is the unit matrix and τ_3 the third Pauli matrix in iso-space. Using these expressions, we go over to the momentum representation:

$$\varphi(p) = \gamma_\mu O_n\chi(p)\,O_n\gamma_\nu J_{\mu\nu}(p)\,\tau_0 + {}^1/_2\,[\gamma_\mu O_n\chi(p)\,O_n - O_n\chi(p)\,O_n\gamma_\mu]\,J_\mu(p)\,\tau_3, \qquad (22a)$$

$$\chi(p) = \gamma_\mu O_n\,\mathrm{tr}\,\varphi(p)\,O_n\gamma_\nu J_{\mu\nu}(p) + {}^1/_2\,[\gamma_\mu O_n\,\mathrm{tr}\,\varphi(p)\,O_n - O_n\,\mathrm{tr}\,\varphi(p)\,O_n\gamma_\mu]\,J_\mu(p). \qquad (22b)$$

Here $\varphi(p)$ and $\chi(p)$ denote the Fourier components of $\varphi(x, x)$ and $\chi(x, x)$ (see (7)), and the momentum integrals J_μ and $J_{\mu\nu}$ are given by the formulas (10).

It follows from the definition (5) that the isotopic structure of the amplitude $\varphi(p)$ is

$$\varphi = \begin{pmatrix} \langle p\bar{p}\rangle & \langle p\bar{n}\rangle \\ \langle n\bar{p}\rangle & \langle n\bar{n}\rangle \end{pmatrix} = \begin{pmatrix} \pi_0^1 & \pi^+ \\ \pi^- & \pi_0^2 \end{pmatrix}. \qquad (23)$$

Comparing this expression with (22a), we conclude that the amplitudes of the charged mesons are in this version of the theory identically equal to zero:

$$\varphi_{\pi^\pm}(p) \equiv 0. \qquad (24)$$

Since S^{χ} and S^{ψ} are diagonal in the isotopic spin indices, this result is preserved even in the higher approximations of the Tamm-Dancoff method.

For the amplitude of the neutral meson with isotopic spin $T = 0$ (reference 4),

$$\varphi_0(p) = (\langle p\bar{p}\rangle + \langle n\bar{n}\rangle)/\sqrt{2} = \mathrm{tr}\,\tau_0\varphi(p)/\sqrt{2}, \qquad (25)$$

we obtain the following system of equations [see formulas (22a,b)]:

$$\sqrt{2}\,\varphi_0(p) = 2\gamma_\mu O_n\chi(p)\,O_n\gamma_\nu J_{\mu\nu}(p),$$
$$\chi(p) = \sqrt{2}\,\gamma_\mu O_n\varphi_0(p)\,O_n\gamma_\nu J_{\mu\nu}(p) + {}^1/_2\sqrt{2}\,[\gamma_\mu O_n\varphi_0(p)\,O_n - O_n\varphi_0(p)\,O_n\gamma_\mu]\,J_\mu(p). \qquad (26)$$

Simplifying it by the method described in the previous section, we obtain

$$\begin{vmatrix} 2(A_\rho)^2(\beta_\rho C + \varepsilon_\rho p^2 D)^2 - 1 & (1-\varepsilon_\rho)\,A_\rho A_\sigma(\beta_\rho C + \varepsilon_\rho p^2 D)\,p_4 B \\ (1-\varepsilon_\sigma)\,A_\rho A_\sigma(\beta_\sigma C + \varepsilon_\sigma p^2 D)\,p_4 B & 2(A_\sigma)^2(\beta_\sigma C + \varepsilon_\sigma p^2 D)^2 - 1 \end{vmatrix} = 0 \qquad (27)$$

(the coefficients A_ρ, β_ρ, and ϵ_ρ and the functions B, C, and D are defined in section 2).

For the pseudoscalar meson we have

$$\beta_\rho = -4, \qquad \varepsilon_\rho = -1, \qquad \beta_\sigma = 2, \qquad \varepsilon_\sigma = -1, \qquad (28)$$

and A_ρ depends on the choice of the version.

The constant $(4\pi/\kappa l)^2$ entering in (27) was also determined by computing the mass of the proton. All the necessary quantities together with the results of the numerical solution of equation (27) for the pseudoscalar meson are given in Table II.

The amplitude of the neutral meson with isotopic spin $T = 1$ (reference 4),

$$\varphi_1(p) = (\langle p\bar{p}\rangle - \langle n\bar{n}\rangle)/\sqrt{2} = \mathrm{tr}\,\tau_3\varphi(p)/\sqrt{2}, \qquad (29)$$

is, according to (22a), given in the form

$$\varphi_1(p) = [\gamma_\mu O_n\chi(p)\,O_n - O_n\chi(p)\,O_n\gamma_\mu]\,J_\mu(p)/\sqrt{2}. \qquad (30)$$

But since $\chi(p) \sim \varphi_0(p)$, the only non-zero Fourier components of $\varphi_1(p)$ are those, whose mass coincides with the mass of the isoscalar meson mentioned earlier. The comparison of the data of Table II with those of Table I shows that the masses of the pseudoscalar mesons are in all versions of the theory lowered, as was to be expected (see the remarks at the end of section 1).

TABLE II

Version	$(4\pi/\varkappa l)^2$	A_ρ	A_σ	M_{π^0}/M_p	Version	$(4\pi/\varkappa l)^2$	A_ρ	A_σ	M_{π^0}/M_p
S	3.345	1	1	0.19 0.30	T	11.209	6	0	<0.01
					A	7.208	4	—2	0.46
V	4.146	—4	2	0,42 0.61	P	2.470	1	—1	0.31 0,42

However, even in this case the disagreement between the theoretical and experimental results in the scalar version amounts to $\sim 25\%$ and even more in the other versions. We obtained a similar result in the calculation of the mass of the K meson.[5]

3. CONCLUSION

The above calculations of the mass spectrum of the mesons show that the scalar version of the nonlinear term is preferable to the others in the first approximation of the Tamm-Dancoff method. It should be kept in mind, however, that it is not obvious that the first approximation is sufficient. The comparison with experiment therefore is rather of a qualitative nature.

Taking account of the isotopic properties improves the results for the mass, but also leads to difficulties with the charged mesons. As long as the Lagrangian (17) is retained, this difficulty can only be removed by a complete change of the isotopic structure of the commutator function $S^{\psi}(q)$.

In conclusion I express my gratitude to Prof. W. Heisenberg and Dr. H. Mitter for making available the numerical calculations omitted in their paper.[2] I. G. Golyak was very helpful in the calculations of Sec. 1.

[1] W. Heisenberg, Revs. Modern Phys. **29**, 269 (1957).

[2] Heisenberg, Kortel, and Mitter, Z. Naturforsch. **10a**, 425 (1955).

[3] R. Ascoli and W. Heisenberg, Z. Naturforsch. **12a**, 177 (1957).

[4] H. A. Bethe and J. Hamilton, Nuovo cimento **4**, 1 (1956).

[5] Ya. I. Granovskii, Уч. зап. Каз. гос. ун-та (Scientific Papers of Kazakh State University) (in press)

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