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CERENKOV RADIATION OF AN ELECTRON MOVING IN A MEDIUM WITH SPATIAL DISPERSION

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An analysis is given of Cerenkov radiation in an isotropic gyrotropic medium, with spatial dispersion taken into account. The angular distribution of the Cerenkov radiation and the emergence of the radiation at the boundary of the medium are considered.

1. INTRODUCTION

DIELECTRICS are usually characterized by a dielectric constant which depends only on frequency ω. However, the electric induction can also depend on the spatial derivatives of the field intensity. For example, if one takes account of the first spatial derivatives it is possible to describe the effects due to natural optical activity. The terms containing derivatives in the expressions which relate the electric field intensity and the electric induction are, generally speaking, of higher order than the electromagnetic field equations. These lead to the new solutions and new waves which have been investigated by a number of authors.[1-3] As has already been noted,[2] one of the possible methods of observing effects related to spatial dispersion is to examine the Cerenkov radiation.

It has been shown in reference 4 that the total energy loss of an electron which moves in an arbitrary medium characterized by spatial dispersion is given by the following expression:

$$F = -\frac{ie^2}{2\pi^2 c^2 v} \times \int (\mathbf{q}\cdot\mathbf{v}) \frac{|v_i b_{ik}^{-1} q_k|^2 - (v_i b_{ik}^{-1} v_k)(q_i b_{ik}^{-1} q_k) + (v_i b_{ik}^{-1} v_k)}{1 - (q_i b_{ik}^{-1} q_k)} d\mathbf{q}, \quad (1.1)$$

where the tensor b_{ik} is expressed in terms of the dielectric permittivity tensor as follows:

$$b_{ik} = q^2 \delta_{ik} - \frac{(\mathbf{q}\cdot\mathbf{v})^2}{c^2} \varepsilon_{ik}. \quad (1.2)$$

The permittivity tensor ϵ_{ik} depends only on frequency (ω) if spatial dispersion is neglected; however, if spatial dispersion is considered ϵ_{ik} also depends on the propagation vector $\mathbf{q}$, i.e.

$$D_i = \varepsilon_{ik}(\omega, \mathbf{q}) E_k. \quad (1.3)$$

We first consider an isotropic medium. In this case Eq. (1.3) assumes the form

$$\varepsilon_{ik}(\omega, q) = \varepsilon_1(\omega, q)\delta_{ik} + \varepsilon_2(\omega, q) q_i q_k / q^2, \quad (1.3a)$$

while the expression for the total energy loss becomes:*

$$F = -\frac{ie^2}{\pi} \int_{-\infty}^{+\infty} \omega\, d\omega \times \int_0^{k_0} k\, dk \frac{v^{-2} - c^{-2}\bar{\varepsilon}(\omega, k^2 + \omega^2/v^2) + \dfrac{\omega^2}{(v^2k^2 + \omega^2)}\varepsilon_2(\omega, k^2 + \omega^2/v^2)}{\bar{\varepsilon}(\omega, k^2 + \omega^2/v^2)\{k^2 + \omega^2[v^{-2} - c^{-2}\varepsilon_1(\omega, k^2 + \omega^2/v^2)]\}}, \quad (1.4)$$

where $\bar{\epsilon} = \epsilon_1 + \epsilon_2$.

It follows from Eq. (1.4) that Cerenkov radiation will be excited at frequency ω if the following condition is satisfied:

$$v > c/n_i(\omega), \quad (1.5)$$

where the quantity $n_i(\omega)$ satisfies the relation

$$n^2(\omega) = \varepsilon_1(\omega, n^2\omega^2/c^2). \quad (1.6)$$

If spatial dispersion of the medium is neglected Eq. (1.6) defines a single function $n(\omega)$. On the other hand, if spatial dispersion is taken into account this equation will in general, have several solutions $n_i = n_i(\omega)$. If ϑ is the angle between the direction of motion of the particle and the radiation direction, since

$$\cos\vartheta_i = c / v n_i(\omega), \quad (1.7)$$

we find that the Cerenkov radiation at frequency ω is distributed over cones with opening angles ϑ_i which are determined from Eq. (1.7). It is of interest to consider the intensity distribution of the Cerenkov radiation over these cones.

2. INTENSITY DISTRIBUTION OF THE CERENKOV RADIATION

Carrying out the integration indicated in Eq. (1.4) with appropriate circuits of the poles in the complex

*The upper limit of the integration over k and k_0 is determined by the range of validity of the macroscopic analysis.

plane for k (cf. reference 5), we find that the energy loss due to Cerenkov radiation in the frequency interval ω, $\omega + d\omega$ is given by the expression

$$dF = \frac{e^2}{c^2} \times \sum_i \left(1 - \frac{c^2}{v^2 n_i^2(\omega)}\right) \left| 1 - \frac{d}{dn_i^2(\omega)} \varepsilon_1\left(\omega, \frac{\omega^2}{c^2} n_i^2(\omega)\right)\right|^{-1} \omega d\omega. \quad (2.1)$$

Eq. (2.1) represents the total intensity of the Cerenkov radiation as a function of frequency; this intensity is a sum of intensities which are distributed over the individual Cerenkov cones defined by Eq. (1.7).

We now examine the different possibilities which arise when spatial dispersion of the medium is considered.

At Cerenkov frequencies far from the resonance frequencies of the medium the solution of Eq. (1.6) can be found through the use of a "direct" dispersion expansion. Limiting ourselves to second spatial derivatives,[2] we have for the transverse waves

$$\mathbf{D} = (\varepsilon_0(\omega) - \alpha(\omega) n^2)\,\mathbf{E} \equiv \varepsilon_1(\omega, n^2\omega^2/c^2)\,\mathbf{E}. \quad (2.2)$$

Hence, in the frequency regions considered here Eq. (1.6) has the single solution

$$n^2(\omega) = \varepsilon_0(\omega)/(1 + \alpha(\omega)), \quad (2.3)$$

and the Cerenkov radiation is distributed over a single cone given by Eq. (1.7). In this frequency region $\alpha(\omega) \ll 1$; thus spatial dispersion of the medium has essentially no effect on the Cerenkov radiation of an electron in these cases. Equation (2.2), which gives the intensity of the Cerenkov radiation at frequency ω, assumes the form

$$dF = \frac{e^2}{c^2} \frac{1 - c^2(1 + \alpha(\omega))/v^2\varepsilon_0(\omega)}{|1 - \alpha(\omega)|} \omega d\omega = \frac{e^2}{c^2} \frac{1 - c^2/v^2 n^2}{2 - \varepsilon_0/n^2} \omega d\omega. \quad (2.4)$$

On the other hand, dispersion becomes important at Cerenkov frequencies close to the resonance frequencies of the medium. In this case, it is necessary to use the "inverse" dispersion expansion in solving Eq. (1.6). If, as in the above paragraph, we limit ourselves to second spatial derivatives,[2] we have

$$\mathbf{E} = \mathbf{D}/\hat{\varepsilon}(\omega, n^2\omega^2/c^2) = [1/\varepsilon_0 + \beta n^2]\,\mathbf{D}. \quad (2.5)$$

Thus, at these frequencies the dispersion equation (1.6) has two roots:

$$n_{1,2}^2(\omega) = -1/2\,\varepsilon_0\beta \pm \sqrt{(2\varepsilon_0\beta)^{-2} + 1/\beta}. \quad (2.6)$$

Ginzburg has shown[2] that when $\beta > 0$, one of the roots of Eq. (2.6) is always smaller than unity and the Cerenkov condition (1.5) may not be satisfied. In this case the Cerenkov radiation is distributed over a single cone. However, when $\beta < 0$, the Cerenkov condition can be satisfied for both roots of (2.6) and the Cerenkov radiation will be distributed over two cones. The total intensity of the Cerenkov radiation is then given by the sum of two terms — the intensities of radiation over these cones:

$$dF = \frac{e^2}{c^2} \sum_{1,2} \frac{1 - c^2/v^2 n_i^2}{|1 + \beta n_i^4|} \omega\, d\omega. \quad (2.7)$$

In accordance with Eq. (2.7), the ratio of these intensities is given by the following expression:

$$\frac{I_2}{I_1} = \left(1 - \frac{c^2}{v^2 n_2^2}\right)|1 + \beta n_1^4| \Big/ \left(1 - \frac{c^2}{v^2 n_1^2}\right)|1 + \beta n_2^4|. \quad (2.8)$$

It follows from Eqs. (2.6) and (2.7) that when $\epsilon_0^2|\beta| \ll 1$, $n_2^2 \gg n_1^2$ and

$$I_2/I_1 \approx \varepsilon_0^2|\beta|/(1 - c^2/v^2 n_1^2) \ll 1. \quad (2.9)$$

It follows from the foregoing that if the condition $\epsilon_0^2|\beta| \ll 1$ is satisfied the Cerenkov radiation is concentrated almost entirely in the first (ordinary) cone. As Ginzburg has shown, in the optical region of the spectrum this condition is satisfied close to the center of an absorption line. Hence, for a non-gyrotropic medium the new Cerenkov radiation could be observed at the center of the absorption line (at wavelengths such that $\Delta\lambda < 5$ A, when $\epsilon_0^2|\beta| \sim 1$). In this case the intensity of the new Cerenkov radiation becomes comparable with the intensity of the usual Cerenkov radiation. However, close to the absorption line one must take account of absorption of the radiation in the medium. Estimates carried out in reference 2 indicate that in actual materials the observation of Cerenkov radiation in the optical region of the spectrum would be possible only in sheets which are approximately 10^{-4} cm thick. Thus the experimental observation of the new Cerenkov radiation in an isotropic medium would be difficult because of absorption in the medium.

In terms of the model proposed in reference 3, the real dielectric constant (at one of the natural frequencies of the medium) assumes the following form when spatial dispersion in an isotropic non-gyrotropic medium is considered ($\mathbf{q} = n\omega\mathbf{s}/c$):

$$\varepsilon_{ik}\left(\omega, \frac{\omega^2}{c^2} n^2\right) = \left(1 - \frac{\omega_0^2}{\omega^2 + \omega_0^2 - \beta_0 - \alpha(\omega/c)^2 n^2}\right)\delta_{ik} - \frac{\alpha_1 q_i q_k}{\omega^2 + \omega_0^2 - \beta_0 - \alpha(\omega/c)^2 n^2}. \quad (2.10)$$

The dispersion relation (1.6) now has two solutions, $n_{1,2}^2$, and the Cerenkov radiation is distributed over two cones. The total intensity of the Cerenkov radiation is again given as the sum of two terms:

$$dF = \frac{e^2}{c^2} \sum_{i=1,2} \left(1 - \frac{c^2}{v^2 n_i^2}\right) \left| 1 + \frac{\alpha}{c^2} \frac{\omega^2}{\omega_0^2} (n_i^2 - 1)^2 \right|^{-1} \omega d\omega. \tag{2.11}$$

We now consider Cerenkov radiation in an isotropic gyrotropic medium characterized by spatial dispersion. Since spatial dispersion is important in frequency regions close to one of the resonance frequencies of the medium we use the "inverse" dispersion expansion:

$$E_i = [(1/\varepsilon_0 + \beta n^2)\,\delta_{ik} + i\gamma\omega e_{ikl}\, n s_l]\, D_k. \tag{2.12}$$

Eq. (1.1), which characterizes the total energy loss of an electron moving in such a medium, assumes the form*

$$F = \frac{ie^2}{\pi} \int_{-\infty}^{+\infty} \int_0^{k_0} \omega\, d\omega\, k\, dk \tag{2.13}$$

$$\times \frac{a + \gamma^2 b}{\left[\frac{v^2\omega^2}{c^2(\omega^2 + k^2 v^2)} - \frac{1}{\varepsilon_0} - \beta \frac{c^2}{\omega^2}\left(\frac{\omega^2}{v^2} + k^2\right)\right]^2 - \gamma^2 \frac{c^2}{v^2}(\omega^2 + k^2 v^2)},$$

where we have introduced the notation

$$\begin{aligned} a &= \left[\frac{1}{\varepsilon_0} + \beta \frac{c^2}{\omega^2}\left(\frac{\omega^2}{v^2} + k^2\right)\right] \\ &\times \left[\frac{v^2\omega^2}{c^2(\omega^2 + k^2v^2)} - \frac{1}{\varepsilon_0} - \beta\frac{c^2}{\omega^2}\left(\frac{\omega^2}{v^2} + k^2\right)\right] \\ &\times \frac{1}{\omega^2 + k^2v^2}\left[\frac{v^2}{c^2} - \frac{1}{\varepsilon_0} - \beta\frac{c^2}{\omega^2}\left(\frac{\omega^2}{v^2} + k^2\right)\right], \\ b &= \frac{c^2}{\omega^2 + k^2v^2}\left\{k^2\left[\frac{v^2}{c^2} - \frac{1}{\varepsilon_0} - \beta\frac{c^2}{\omega^2}\left(\frac{\omega^2}{v^2} + k^2\right)\right]\right. \\ &\left. - \frac{\omega^2}{v^2}\left[\frac{1}{\varepsilon_0} + \beta\frac{c^2}{\omega^2}\left(\frac{\omega^2}{v^2} + k^2\right)\right]\right\}. \end{aligned} \tag{2.14}$$

Examination of Eq. (2.13) indicates that Cerenkov radiation at frequency ω is excited in an isotropic gyrotropic medium with spatial dispersion if Eq. (1.5) is satisfied; the $n_i(\omega)$ in this equation are determined from the dispersion equation

$$\left(\frac{1}{n^2} - \frac{1}{\varepsilon_0} - \beta n^2\right)^2 - \gamma^2\omega^2 n^2 = 0, \quad n^2\frac{\omega^2}{c^2} \equiv \frac{\omega^2}{v^2} + k^2. \tag{2.15}$$

In general, Eq. (2.15) yields three different values of $n_i^2(\omega)$ so that in the case being considered the Cerenkov radiation is distributed over the surfaces of three cones with opening angles ϑ_i, which are determined from Eq. (1.7). In this case the total intensity of Cerenkov radiation at frequency ω is given by the expression:

$$dF = -e^2 \sum_{i=1,2,3} \omega\, d\omega \frac{a_i + \gamma^2 b_i}{|2(\beta + n_i^{-4})(n_i^{-2} - 1/\varepsilon_0 - \beta n_i^2)\, c^2/\omega^2 + \gamma^2 c^2|}. \tag{2.16}$$

*It should be kept in mind that Eq. (2.13) gives the correct value of the loss due to Cerenkov radiation only for those frequency regions in which the inverse dispersion relation (2.12) is valid (cf. reference 3).

In frequency regions in which $\gamma\omega\epsilon_0 n \ll 1$ and $\beta n^2\epsilon_0 \ll 1$, Eq. (2.15) has two roots which are approximately equal

$$n_{1,2}^2 = n_0^2(1 \pm \gamma\omega n_0^3) \tag{2.17}$$

and a third root which is larger

$$n_3^2 = 1/n_0^4(\gamma^2\omega^2 - 2\beta/n_0^2), \tag{2.18}$$

where $n_0^2 = \epsilon_0$. Consequently, if the radiation intensities for the cones are to be of the same order of magnitude the following condition must be satisfied:

$$n_0^6(\gamma^2\omega^2 - 2\beta/n_0^2) \approx 1. \tag{2.19}$$

In the optical region of the spectrum $\gamma^2\omega^2 \approx 10^{-5}$ and $\beta \approx 10^{-6}$; thus, $n_0^2 = 45$. According to reference 2, this condition is satisfied at wave lengths such that $\Delta\lambda \lesssim 100\,\text{A}$ as measured from the center of the absorption line. It is apparent that absorption does not play an important role in a gyrotropic medium and that the experimental observation of the new Cerenkov waves may be possible.

3. EMERGENCE OF CERENKOV RADIATION THROUGH THE BOUNDARY OF THE MEDIUM

To determine the intensity of the Cerenkov radiation which emerges from a medium it is necessary to consider effects at the boundary. The question of boundary condition then arises. It is apparent that the usual continuity conditions $D_{2n} = D_{1n}$, $H_{2n} = H_{1n}$, $E_{2t} = E_{1t}$ and $H_{2t} = H_{1t}$ still hold. However, new wave solutions are possible when spatial dispersion is taken into account and the usual boundary conditions are found to be inadequate for determining the amplitudes of these waves. As an additional boundary condition Pekar[1] has introduced the condition that the dipole moment per unit volume, due to the excitation of exciton waves in the medium, must vanish

$$4\pi\mathbf{P} = \mathbf{D} - \varepsilon'\mathbf{E} = \sum_{i=1,2} (n_i^2 - \varepsilon')\,\mathbf{E} = 0, \tag{3.1}$$

where the quantity ϵ' takes account of the contribution due to the other resonance frequencies of the medium. The boundary conditions in (3.1) can be used to estimate the intensity of the radiation which emerges from the medium.

Suppose that at the interface between an isotropic non-gyrotropic medium and a vacuum, a plane wave is incident on the medium at an angle $\vartheta_0 < \pi/2$. Two reflected waves arise, corresponding to the two refractive indices $n_1(\omega)$ and $n_2(\omega)$. Suppose that the index associated with the incident wave is $n_1(\omega)$. From the condition that the phases must be equal it is easy to show that the following relations obtain between the refracted and reflected waves:

$$\vartheta_0 = \vartheta_1, \ \sin\vartheta_2 / \sin\vartheta_0 = n_1(\omega)\sin\vartheta / n_2(\omega)\sin\vartheta_0 = n_1(\omega), \tag{3.2}$$

where ϑ is the angle of refraction and ϑ_1 and ϑ_2 are the angles of reflection. The boundary conditions (3.1) can be used to determine the amplitudes of the refracted and reflected waves. If the electric vector of the incident wave is perpendicular to the plane of incidence the amplitude of the refracted wave E is related to the amplitude of the incident wave E_0 by the expression

$$\frac{E}{E_0} = \frac{2n_1\cos\vartheta_0(n_1^2 - n_2^2)}{\cos\vartheta(n_1^2 - n_2^2) + n_1\cos\vartheta_0\,(\varepsilon' - n_2^2) - n_2\cos\vartheta_2(\varepsilon' - n_1^2)}. \tag{3.3}$$

If the incident wave is associated with the index $n_2(\omega)$, we interchange n_1 and n_2 in Eqs. (3.2) and (3.3).

The results obtained above can be used to choose the most convenient experimental geometry for studying these features of Cerenkov radiation in media characterized by spatial dispersion.

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