PARENTAGE COEFFICIENTS IN THE GENERALIZED NUCLEAR MODEL

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We consider the construction of a wave function of a system of n nucleons, with prescribed values of isobarin spin T, from the wave functions of individual states N_1, N_2, ... N_n that are not coupled vectorially. The number of independent states with a given T is determined by algebraic methods. The problem is primarily of interest in relation to the unified model of a light nucleus.

NUMEROUS experimental data[1] evidence that the generalized model of a nucleus with a strong coupling between the particle and the surface holds not only for heavy nuclei, but also for elements between O^{16} and Ca^{40} (1d and 2s shells), i.e., where the isobaric spin is a good quantum number for the lower states. In connection with this, it becomes necessary to evaluate for the Nilsson model[2] in this region the parentage coefficients[3] necessary to calculate the matrix moments of the operators $F = \Sigma f_i$ or $G = \Sigma g_{if}$.

In this paper we make a general analysis of this problem and derive a general formula for the parentage coefficients.

1. NUMBER OF INDEPENDENT STATES

The nucleon in the nucleus is characterized by the charge and by four quantum numbers, for example nljΩ, where Ω is the projection of the nucleon momentum on the symmetry axis of the nucleus. The particular form of this set of numbers is immaterial in our problem. We denote this set by N.

The wave function of n particles is

$$|N_1 N_2 \ldots N_n \alpha T T_Z\rangle = \Psi_{T, T_Z}, \qquad (1.1)$$

where T is the isobaric spin, $T_Z = Z - \frac{1}{2}A$ is antisymmetric over all particles if the charge coordinate of the nucleon is introduced.

It is easy to see that by changing the order of the nucleon coupling we can obtain other independent states from (1.1). Such states are identified by the symbol α. The number of such independent states is determined by the theory of permutation groups[4] in the following manner. The antisymmetric function Ψ is constructed from the spin-orbit functions $\Phi^{[\lambda]}(r)$ and the charge functions $\Gamma^{[\tilde{\lambda}]}(r)$, where λ is the symbol of the Young tableau, r is the Yamanouchi symbol,[4] while $[\tilde{\lambda}]$ and $[\tilde{r}]$ are the symbols for the tableaus that are conjugate[5] to $[\lambda]$ and r in the following manner

$$\Psi_{T, T_Z} = (f^{[\lambda]})^{-1/2} \sum_r \Phi^{[\lambda]}(r)\, \Gamma^{[\tilde{\lambda}]}_{T, T_Z}(\tilde{r}), \qquad (1.2)$$

where $f^{[\lambda]}$ is the dimensionality of the representation of the group S_n of permutation of n numbers, with $[\lambda]$ determining the isobaric spin T.[5] It is necessary to determine how many independent states $\Phi^{[\lambda]}(r)$ exist with given $[\lambda]$ and r.

We first consider the case when all N_i are different (at least one number out of four different for each compared pair).

We introduce

$$\Phi_0 = |N_1(1)\, N_2(2) \ldots N_n(n)\rangle, \qquad (1.3)$$

i.e., Φ_0 is simply the product of individual functions. All the functions $P\Phi_0$ for all n! permutations of the number of states P are orthogonal to each other. From these we construct the functions

$$\omega^{[\lambda]}_{rs}\Phi_0, \qquad (1.4)$$

where $\omega^{[\lambda]}_{rs}$ are the normalized orthogonal Young operators,[4] $[\lambda]$ is the symbol for the Young tableau, while r and s are the Yamanoochi symbols. The symbols $[\lambda]$ and r represent the permutation properties with respect to permutation of particles, while the symbols $[\lambda]$ and s pertain to permutation of the number of states.

All the states $\omega^{[\lambda]}_{rs_i}\Phi_0$ ($i = 1, 2, f^{[\lambda]}$) are orthogonal to each other. In addition, since

$$P\omega^{[\lambda]}_{rs} = \sum_t \theta^{[\lambda]}_{rt}(P)\, \omega^{[\lambda]}_{rt}, \qquad (1.5)$$

(where $\theta^{[\lambda]}_{rt}(T)$ is the matrix of the irreducible orthogonal representation[4] of the group S_n) any state with specified $[\lambda]$ and r can be expressed in terms of these states. Thus, if all N_i are dif-

ferent, the number of independent states with specified T is $f^{[\lambda]}$.

Now let there be one pair of identical N_i, for example, $N_1 = N_2$. Then

$$\Phi_1 = |N_1(1)\,N_1(2)\,N_3(3)\ldots N_n(n)\rangle \qquad (1.6)$$

has the same properties with respect to permutation as

$$[\varepsilon + P_{12}]\,\Phi'_0, \qquad (1.7)$$

where ϵ is the unit of the group; P_{12} is the permutation of the number of states 1 and 2; $[\epsilon + P_{12}] = 2e^{[2]}$ is the symmetrizer with respect to the numbers 1 and 2, and the function Φ'_0 has the same properties with respect to permutation as Φ_0.

Thus, the problem reduces to the following: how many of the operators

$$\omega^{[\lambda]}_{rs_i} e^{[2]}, \quad i = 1, 2 \ldots f^{[\lambda]}, \qquad (1.8)$$

are linearly independent?

The problem is solved by successive expansion of $\omega^{[\lambda]}_{rs}$ in idempotent operators of the Young tableaus with an ever-diminishing number of cells[4] until $\omega^{[\lambda]}_{rs}$ assumes the form of an operator product on the right end of which will be either $e^{[11]}$ — the anti-symmetrizer over numbers 1 and 2, or $e^{[2]}$:

$$e^{[11]}e^{[2]} = 0, \quad e^{[2]}e^{[2]} = e^{[2]}.$$

Thus

$$\omega^{[\lambda]}_{rs_i} e^{[2]} = \omega^{[\lambda]}_{rs_i},$$

if in a table of s_i the numbers 1 and 2 are in the first row (i.e., $[\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_m]$ should have $\lambda_1 = 2$), and

$$\omega^{[\lambda]}_{rs_i} e^{[2]} = 0$$

in the opposite case. The number of non-vanishing operators $\omega^{[\lambda]}_{rs_i} e^{[2]}$, as can be readily concluded from the above, is $f^{[\lambda^*]}$, where $[\lambda^*] = [\lambda_2, \ldots \lambda_m]$ is the Young tableau without the first line. All are independent of each other.

Finally, let us consider the case of k identical pairs N from a total number and $n \geq 2k$. It is necessary to determine how many independent operators will be among

$$\omega^{[\lambda]}_{rs_i} e^{[2]_1} e^{[2]_2} \ldots e^{[2]_k}, \quad i = 1, 2 \ldots f^{[\lambda]}, \qquad (1.9)$$

where $e^{[2]_j}$ is the symmetrizer over the j-th pair of states. Making use of the results of the foregoing analysis and solving the problem by analogy, we find that the number of independent operators is $f^{[\lambda^k]}$ where $[\lambda^k]$ is obtained from $[\lambda]$ by removing the k upper rows, each of which consist of two cells. The number of independent operators equals the number of independent states.

The method of constructing the function $\Phi^{[\lambda]}_{rs} = \omega^{[\lambda]}_{rs}\Phi$ (where Φ is Φ_0 or Φ_1, etc.) i.e., the Yamanouchi symbol s, is determined, as can be readily seen, by all the intermediate particle couplings over T:

$$|(N_1N_2)\,T_{12}, N_3, T_{123}, \ldots, N_n, TT_z\rangle = (f^{[\lambda]})^{-1/2} \sum_r \omega^{[\lambda]}_{rs}\Phi\,\Gamma^{[\lambda]}_{\tilde{r}}(\tau(1), \tau(2) \ldots, \tau(n)\,TT_z). \qquad (1.10)$$

Actually, the transition from T_{12} to T_{123} etc. describes the successive filling of the Young tableau by numbers; this determines the Yamanouchi symbol.

2. PARENTAGE COEFFICIENTS AND THEIR CALCULATION

The parentage coefficients make it possible to express the complete asymmetric wave function of n nucleons in the form of a linear combination of antisymmetric wave functions of $n-1$ particles, vectorially coupled through the isobaric spin with the wave function of the n-th particle. In the spin-orbit portion of the wave function, there is no vector addition:

$$\begin{aligned}|N_1N_2\ldots N_n\alpha TT_Z\rangle &= \sum_{\alpha'T's} \langle N_1N_2\ldots N_n\alpha T | N_1N_2\ldots N_{s-1}N_{s+1}\ldots N_n\alpha'T'\rangle \\ &\times |N_1N_2\ldots N_{s-1}N_{s+1}\ldots N_n\alpha'T',\ N_s(n):TT_Z\rangle, \\ &|N_1N_2\ldots N_{s-1}N_{s+1}\ldots N_n\alpha'T',\ N_s(n):TT_Z\rangle \\ &= \sum_{T'_z\tau_z(n)} (T'T'_Z, {}^1/_2\tau_Z(n)\,|\,T'^1/_2TT_Z) \qquad (2.1) \\ &\times |N_1N_2\ldots N_{s-1}N_{s+1}\ldots N_n\alpha'T'T'_Z\rangle |N_s{}^1/_2\tau_Z(n)\rangle.\end{aligned}$$

Here $\langle N_1N_2\ldots N_n\alpha T\,|\,N_1\ldots N_{S-1}N_{S+1}\ldots N_n\alpha' T'\rangle$ is the parentage coefficient. We shall use henceforth a shorter symbol:

$$\langle N_1N_2\ldots N_n\alpha T\,|\,N_1\ldots N_{s-1}N_{s+1}\ldots N_n\alpha'T'\rangle \equiv \langle n\alpha T\,|\,(n-1)_s\alpha'T'\rangle; \qquad (2.2)$$

$\tau_Z(m)$ in formula (2.1) is the projection of the isobaric spin of the n-th nucleon. $(T'T'_Z, \frac{1}{2}\tau_Z(m)\,|\,T'\frac{1}{2}TT_Z)$ is the Clebsch-Gordan coefficient.

To calculate the genealogical coefficients we use the Redmond method.[6] If we take some state $|N_1\ldots N_{i-1}N_{i+1}\ldots N_n\alpha''T'',\ N_i(n):TT_Z\rangle$ from the right half of formula (2.1) as the initial state, and act on it with the antisymmetrizing operator $\hat{O} = 1 - \sum_{k=1}^{n-1} P_{kn}$, we obtain, with accuracy to within a normalizing factor Q, the antisymmetric state $|N_1N_2\ldots N_n\alpha TT_Z\rangle$. The operator P_{kn} is the operator of

TABLE I. Parentage coefficients for n = 3

$\langle N_1N_2N_3\alpha T \mid (2)_sT',\ N_s\rangle$

T'	0			1		
αT \ s	1	2	3	1	2	3
$^1/_2\,a$	$1/2\sqrt{3}$	$-1/\sqrt{3}$	$1/2\sqrt{3}$	$-1/2$	0	$1/2$
$^1/_2\,b$	$1/2$	0	$-1/2$	$1/2\sqrt{3}$	$1/\sqrt{3}$	$1/2\sqrt{3}$
$^3/_2$				$1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{3}$

$\langle N_1N_1N_2\,{}^1/_2 \mid N_1N_2\,0,\ N_1\rangle = -1/\sqrt{6},\ \langle N_1N_1N_2\,{}^1/_2 \mid N_1N_2\,1,\ N_1\rangle = -1/\sqrt{2},$
$\langle N_1N_1N_2\,{}^1/_2 \mid N_1N_1\,0,\ N_2\rangle = 1/\sqrt{3}$

permutation of the spin, isobaric, and ordinary coordinates of the k-th and n-th nucleons. Multiplying (2.2) by $|N_1 \dots N_{S-1}N_{S+1} \dots N_n\alpha' T',\ N_S(n):TT_Z\rangle$* and integrating over the corresponding variables, we get

$$\begin{aligned} &Q\langle n\alpha T \mid (n-1)_s\alpha' T'\rangle \\ &= \langle (n-1)_s\alpha' T',\ N_s(n):TT_Z \mid \hat{O} \mid (n-1)_i\alpha''\ T'',\ N_i(n):TT_Z\rangle \\ &= \delta(is)\,\delta(\alpha'\alpha'')\,\delta(T'T'') - \sum_{k=1}^{n-1} \langle (n-1)_s\alpha' T', \qquad (2.3) \\ &\quad N_s(n):TT_Z \mid P_{kn} \mid (n-1)_i\alpha'' T'',\ N_i:TT_Z\rangle. \end{aligned}$$

Consider the matrix element of the permutation $P_{n-1,n}$. This is best performed if we use the parentage coefficients to separate also the (n − 1)-st particle:

$$\begin{aligned} &P_{n-1,n} \mid (n-1)_i\alpha'' T'',\ N_i(n):TT_Z\rangle \\ &= P_{n-1,n} \sum_{\alpha''' T'''_r} \langle (n-1)_i\alpha'' T'' \mid (n-2)_{ir}\alpha''' T'''\rangle \\ &\times \mid (n-2)_{ir}\alpha''' T''',\ N_r(n-1):T''';\ N_i(n):TT_Z\rangle \\ &= \sum \langle (n-1)_i\alpha'' T'' \mid (n-2)_{ir}\alpha''' T'''\rangle \qquad (2.4) \\ &\times \mid (n-2)_{ir}\alpha''' T''',\ N_r(n):T'';\ \ N_i(n-1):TT_Z\rangle \\ &= \sum_{\alpha''' T''' r T'} \langle (n-1)_i\alpha'' T'' \mid (n-2)_{ir}\alpha''' T'''\rangle \\ &\times \mid (n-2)_{ir}\alpha''' T''',\ N_i(n-1):T';\ N_r(n):TT_Z\rangle \\ &\times U\left(\tfrac{1}{2}T'''T\tfrac{1}{2};\ T''T'\right)(-1)^{T-T'-T''+T'''}. \end{aligned}$$

The last transition is carried out by introducing the Racah coefficients $U(\frac{1}{2}T'''T\frac{1}{2};\ T''T')$, which are connected with the Racah coefficients W as follows

$$U(abcd;ef) = \sqrt{(2e+1)(2f+1)}\,W(abcd;ef).$$

It is now easy to obtain from (2.4) the permutation matrix element $P_{n-1,n}$, which enters into the right half of (2.3). If all N_S are different, we obtain for $s \neq i$, by using the orthogonality of the single-nucleon functions with different N_S,

$$\begin{aligned} &\langle (n-1)_s\alpha' T',\ N_s(n):TT_Z \mid P_{n-1,n} \mid (n-1)_i\alpha'' T'',\ N_i(n):TT_Z\rangle \\ &= \sum_{\alpha''' T'''} \langle (n-1)_s\alpha' T' \mid (n-2)_{is}\alpha''' T'''\rangle \\ &\times \langle (n-1)_i\alpha'' T'' \mid (n-2)_{is}\alpha''' T'''\rangle \qquad (2.5) \\ &\times (-1)^{T-T'-T''+T'''}U(^1/_2T'''T^1/_2;\ T''T'). \end{aligned}$$

When s = r, the matrix element is obviously equal to zero.

From the antisymmetry of the functions that determine the matrix elements of the operators P_{kn} in (2.3), it follows with respect to the permutation of the particles 1, 2, . . . , n − 1, that all the matrix elements in (2.3) are identical and have the value given by Eq. (2.5).

We thus obtain for the case when all N_S are different

$$\begin{aligned} Q\langle n\alpha T \mid (n-1)_s\alpha' T'\rangle &= \delta(is)\,\delta(\alpha'\alpha'')\,\delta(T'T'') - (n-1) \\ &\times (1-\delta(is)) \sum_{\alpha''' T'''} \langle (n-1)_s\alpha' T' \mid (n-2)_{is}\alpha''' T'''\rangle \\ &\times \langle (n-1)_i\alpha'' T'' \mid (n-2)_{is}\alpha''' T'''\rangle \qquad (2.6) \\ &\times (-1)^{T-T'-T''+T'''}U(^1/_2T'''T^1/_2;\ T''T'). \end{aligned}$$

To determine the normalizing factor Q, we multiply (2.2) by its complex conjugate and integrate over the corresponding variables:

$$\begin{aligned} Q^2 &= \langle (n-1)_i\alpha'' T'',\ N_i(n):TT_Z \mid \hat{O} \mid (n-1)_i\alpha'' T'',\ N_i(n):TT_Z\rangle \\ &- \sum_{k=1}^{n-1} \langle P_{kn}\{(n-1)_i\alpha'' T'',\ N_i(n):TT_Z\} \qquad (2.7) \\ &\times \mid \hat{O} \mid (n-1)_i\alpha'' T'',\ N_i(n):TT_Z\rangle. \end{aligned}$$

From calculations analogous to the proceeding ones, we find that $Q^2 = n$ for the case when all N_S are different.

According to the Pauli principle, the same N_S can be found among $N_1, N_2, \dots, N_n$ not more than twice. Consequently, the most general case is that when the $N_1N_2 \dots N_n$ contains k pairs of equal N, i.e., when the wave function of the n nucleons has the form

TABLE II. Parentage coefficients for n = 4

$\langle N_1N_2N_3N_4\alpha\, T \mid (3)_s\, \alpha' T',\ N_s\rangle$

$T'\alpha'$	$^1/_2a$				$^1/_2b$				$^3/_2$			
s / T_α	1	2	3	4	1	2	3	4	1	2	3	4
$0a$	1/2	$-1/4$	$-1/4$	1/2	0	$\sqrt{3}/4$	$-\sqrt{3}/4$	0	·	·	·	·
$0b$	0	$-\sqrt{3}/4$	$-\sqrt{3}/4$	0	$-1/2$	$-1/4$	1/4	1/2	·	·	·	·
$1a$	$-1/\sqrt{6}$	$1\sqrt{6}$	$-1/\sqrt{6}$	0	$-1/3\sqrt{2}$	$1/3\sqrt{2}$	$-1/3\sqrt{2}$	0	1/6	$-1/6$	1/6	1/2
$1b$	0	1/4	$-1/4$	1/2	$1/2\sqrt{3}$	$-\sqrt{3}/4$	$1/4\sqrt{3}$	0	$1/\sqrt{6}$	0	$-1\sqrt{6}$	0
$1c$	$-1/2\sqrt{3}$	$-1/4\sqrt{3}$	$1/4\sqrt{3}$	0	1/3	$-1/12$	$-5/12$	1/2	$-1/3\sqrt{2}$	$-\sqrt{2}/3$	$-1/3\sqrt{2}$	0
2									$-1/2$	1/2	$-1/2$	1/2

$\langle N_1N_1N_2N_2\,0 \mid N_1N_2N_2\,{}^1/_2,\ N_1\rangle = 1/\sqrt{2}$; $\langle N_1N_1N_2N_2\,0 \mid N_1N_1N_2\,{}^1/_2,\ N_2\rangle = 1/\sqrt{2}$;

$\langle N_1N_1N_2N_3\,0 \mid N_1N_2N_3\,{}^1/_2\,a,\ N_1\rangle = (1/2\sqrt{2})$; $\langle N_1N_1N_2N_3\,0 \mid N_1N_2N_3\,{}^1/_2\,b,\ N_1\rangle = \sqrt{3}/2\sqrt{2}$;

$\langle N_1N_1N_2N_3\,0 \mid N_1N_1N_3\,{}^1/_2,\ N_2\rangle = {}^1/_2$; $\langle N_1N_1N_2N_3\,0 \mid N_1N_1N_2\,{}^1/_2,\ N_3\rangle = {}^1/_2$;

$\langle N_1N_1N_2N_3\,1 \mid N_1N_2N_3\,{}^1/_2\,a,\ N_1\rangle = (1/2\sqrt{2})$; $\langle N_1N_1N_2N_3\,1 \mid N_1N_2N_3\,{}^1/_2\,b,\ N_1\rangle = -1/2\sqrt{6}$;

$\langle N_1N_1N_2N_3\,1 \mid N_1N_1N_3\,{}^1/_2,\ N_2\rangle = -{}^1/_2$; $\langle N_1N_1N_2N_3\,1 \mid N_1N_1N_2\,{}^1/_2,\ N_3\rangle = 1/2$;

$\langle N_1N_1N_2N_3\,1 \mid N_1N_2N_3\,{}^3/_2,\ N_1\rangle = 1/\sqrt{3}$

$$\mid N_1N_1N_2N_2 \ldots N_kN_kN_{k+1}N_{k+2} \ldots N_{n-k}\alpha TT_Z\rangle.$$

The formula for the parentage coefficients is derived for such a state of a system of nucleons in the same way as (2.6). We finally obtain

$$\langle n\alpha T \mid (n-1)_s\,\alpha'T'\rangle = Q^{-1}\Big\{\delta_{is}\delta_{\alpha'\alpha''}\,\delta_{T'T''} - (n-1) \times \Big(1 - \delta_{is} + \sum_{j=1}^{k}\delta_{ij}\delta_{sj}\Big) \sum_{\alpha'''T'''} \langle (n-1)_s\,\alpha'T' \mid (n-2)_{is}\,\alpha'''T'''\rangle \times \langle (n-1)_i\,\alpha''T'' \mid (n-2)_{is}\,\alpha'''T'''\rangle . \times (-1)^{T-T'-T''+T'''} U\,({}^1/_2T'''T^1/_2;\ T''T')\Big\};$$

$$Q^2 = n\Big[1 - (n-1)\sum_{\alpha'''T'''}\sum_{j=1}^{k}(-1)^{T-2T''+T'''} U\,({}^1/_2T'''T^1/_2;\ T''T'')\delta_{ij} \times \langle (n-1)_j\alpha''T'' \mid (n-2)_{jj}\,\alpha'''T'''\rangle^2\Big]. \qquad (2.8)$$

In the case of a purely proton or purely neutron configuration, i.e., in the case of maximum isobaric spin, the wave function is merely the Slater determinant and it is obvious that

$$\langle n,\ T = \frac{n}{2} \Big| (n-1)_s,\ T' = \frac{n-1}{2}\rangle = (-1)^{n+s}(n)^{-1/2}.$$

Not all the states $|n\alpha TT_Z\rangle$, obtained with the aid of all possible initial states, are linearly independent. It is necessary to select from among them the linearly-independent and the orthogonal states. This selection is made easier by the fact that if all N_S are different, the initial states $|(n-1)_i\alpha' T'\rangle$ and $|(n-1)_i\alpha'' T''\rangle$ lead to mutually orthogonal states of the n nucleons. It is also useful to know the total number of linearly independent states, which we have determined by algebraic analysis. A unique construction of the wave function is possible only from experimental data, but for the cases of practical importance, when all the nucleons in the configuration form pairs, or else there are one, two, or three nucleons that form no pairs, the matter is rather simple, since the number of linearly-independent states does not exceed two.

Tables I and II give the parentage coefficients for n = 3 and 4. For n = 3 our coefficients coincide with those of Yoshida,[3] whose results, however, are incomplete. The calculations were carried out with the aid of formulas (2.6), (2.7), and (2.8) and can be extended without difficulty to include the case of more particles.

[1] A. E. Litherland et al., Phys. Rev. **102**, 208 (1956). Can. J. Phys. **36**, 378 (1958); **35**, 1057 (1957). G. Racavy, Nucl. Phys. **4**, 289 (1957).

[2] S. G. Nilsson, Dan. Mat. Fys. Medd. **29**, No. 16, 1955.

[3] S. Yoshida, Progr. Theoret. Phys. **12**, 141 (1954).

[4] D. E. Rutherford, Substitutional Analysis, Edinburgh, 1948. A. H. Jahn, Phys. Rev. **96**, 989 (1954).

[5] A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. **A214**, 515 (1952).

[6] P. J. Redmond, Proc. Roy. Soc. **A222**, 94 (1954). A. Hassit, Proc. Roy. Soc. **A229**, 110 (1955).

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