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SYMMETRY GROUP OF THE ISOTROPIC OSCILLATOR

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It is shown that the determination of the isotropic oscillator group by means of infinitesimal operators and generating elements, given earlier by the author, and the determination by means of canonical transformations, given by Baker, are equivalent. The explicit form of the unitary operators of the group is considered and their relation to the Green's function for the oscillator is demonstrated.

We consider an n-dimensional isotropic oscillator and we choose the system of units in such a way that the frequency, Planck's constant, and the mass are all equal to unity. Then the energy operator for the system has the form

$$H = \frac{1}{2} \sum_{k=1}^{n} (p_k^2 + x_k^2), \tag{1}$$

while the energy levels of the sytem

$$E_m = m + n/2 \quad (m = 0, 1, 2, \ldots) \tag{2}$$

are degenerate, with this degeneracy not being accounted for solely by the spherical symmetry of the system.

The symmetry group of the isotropic oscillator which explains the additional degeneracy of the energy levels and which is closely connected with the symmetry between the coordinates and momenta of the system was originally investigated by the author.[1,2] The group of unitary operators commuting with the Hamiltonian was determined by means of generating operators (rotations and one-dimensional Fourier transformations), all possible products of which from the group. Moreover, the infinitesimal operators of the group were obtained and the commutation relations between them were studied.

In Baker's paper[3] the symmetry group of the oscillator was investigated from a somewhat different point of view; the elements of the group were defined by means of a canonical transformation which leaves the Hamiltonian invariant. It was also shown that the symmetry group of the n-dimensional isotropic oscillator is isomorphic with the n-dimensional unitary group.

The relation between the two points of view in the case of the two dimensional oscillator was partially discussed by Alliluev.[4] It turned out that in this case in order to explain the additional degeneracy it is sufficient to consider the unimodular unitary group (which is isomorphic to the group of three dimensional rotations), whose infinitesimal operators are linear combinations of the corresponding operators of references 1 and 2.

We shall show here that the two methods of determining the symmetry group are equivalent in the general case.

We shall first of all establish that the commutation relations between the infinitesimal operators

$$\begin{aligned} H_x &= p_x^2 + x^2; & M_x &= yp_z - zp_y, & N_x &= p_y p_z + yz, \\ H_y &= p_y^2 + y^2, & M_y &= zp_x - xp_z, & N_y &= p_z p_x + zx, \\ H_z &= p_z^2 + z^2, & M_z &= xp_y - yp_x, & N_z &= p_x p_y + xy, \end{aligned} \tag{3}$$

which were obtained in references 1 and 2, coincide with the commutation relations between the infinitesimal matrices of the three dimensional linear unitary group. Indeed, the matrix of an infinitesimal linear unitary transformation may be written in the form

$$U = I + i\varepsilon L, \tag{4}$$

where I is the unit matrix, L is an arbitrary Hermitean matrix, and ϵ is the smallness parameter. It may be seen from this that as linearly independent infinitesimal matrices we can choose the n^2 matrices M^{ij}, N^{ij} with the following elements

$$M_{rs}^{ij} = i(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}), \quad N_{rs}^{ij} = \delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}. \tag{5}$$

The commutation relations between these matrices are of the form

$$\begin{aligned} [M^{ij}, M^{kl}] &= iM^{il}\delta_{jk} - iM^{ik}\delta_{jl} - iM^{jl}\delta_{ik} + iM^{jk}\delta_{il}, \\ [M^{ij}, N^{kl}] &= iN^{il}\delta_{jk} + iN^{ik}\delta_{jl} - iN^{jl}\delta_{ik} - iN^{jk}\delta_{il}, \\ [N^{ij}, N^{kl}] &= -iM^{il}\delta_{jk} - iM^{ik}\delta_{jl} - iM^{jl}\delta_{ik} - iM^{jk}\delta_{il}. \end{aligned} \tag{6}$$

If we now set $n = 3$

$$N^{11} = H_x, \quad N^{12} = N_z, \quad M^{12} = -M_x,$$
$$N^{22} = H_y, \quad N^{23} = N_x, \quad M^{23} = -M_y, \qquad (7)$$
$$N^{33} = H_z, \quad N^{31} = N_y, \quad M^{31} = -M_z,$$

we obtain the commutation relations between the operators (3) which were given in reference 2.* Thus, the symmetry group determined in references 1 and 2 is isomorphic to the unitary group, and consequently is also isomorphic to the symmetry group discussed by Baker.† The generalization to the case $n > 3$ is trivial.

We now show that the two groups are equivalent, i.e., that the unitary operators which carry out Baker's canonical transformations coincide with the operators discussed in references 1 and 2.

Baker[3] introduces the non-Hermitian operators

$$a_k = (x_k + ip_k)/\sqrt{2}, \quad a_k^+ = (x_k - ip_k)/\sqrt{2}, \qquad (8)$$

while the unitary matrix U corresponds to the canonical transformation

$$a_k' = \sum_{l=1}^{n} U_{kl} a_l, \quad \bar{a}_k'^+ = \sum_{l=1}^{n} U_{kl}^* a_l^+. \qquad (9)$$

It may be easily seen that such a transformation leaves invariant the operator H and the commutation relations between the operators a_k and a_k^+. To construct the unitary operator which corresponds to this canonical transformation we return to the variables x_k, p_k. On utilizing (8) and (9), we obtain

$$x_k' = \sum_{l=1}^{n} (A_{kl} x_l - B_{kl} p_l), \quad p_k' = \sum_{l=1}^{n} (B_{kl} x_l + A_{kl} p_l), \qquad (10)$$

$$x_k = \sum_{l=1}^{n} (A_{lk} x_l' + B_{lk} p_l'), \quad p_k = \sum_{l=1}^{n} (-B_{lk} x_l' + A_{lk} p_l'), \qquad (11)$$

where

$$U_{kl} = A_{kl} + iB_{kl} \qquad (12)$$

and A_{kl}, B_{kl} are real. The unitary operator which corresponds to this transformation may be easily obtained in integral form; its kernel $f_U(x_1, x_2, \ldots, x_n; x_1', x_2', \ldots, x_n')$ will be an eigenfunction of the operators x_k in the x'-representation corresponding to the eigenvalues x_k, while f_U^* will be an eigenfunction of the operators x_k' in the x-representation corresponding to the eigenvalues x_k'. This condition determines the kernel f_U up to a factor independent of x_k, x_k'.

We consider a special case of the transformation (10), (11), when only one pair of coordinates and momenta is transformed. In this case we may without loss of generality consider the one-dimensional oscillator. Then $U = \exp i\varphi$, $A = \cos\varphi$, $B = \sin\varphi$ and the kernel of the corresponding integral operator satisfies the following equations

$$-i\sin\varphi \cdot \partial f/\partial x + \cos\varphi \cdot xf = x'f,$$
$$i\sin\varphi \cdot \partial f^*/\partial x' + \cos\varphi \cdot x'f = xf. \qquad (13)$$

From this we obtain

$$f_\varphi(x, x') = C(\varphi)\exp[ixx'/\sin\varphi - (i/2)(x^2 + x'^2)\cot\varphi]. \qquad (14)$$

The coefficient $C(\varphi)$ may be determined by requiring that the operator acting on the ground state wavefunction $\exp(-x^2/2)$ should leave it unchanged. Finally we have

$$f_\varphi(x, x') = (2\pi\sin\varphi)^{-1/2}\exp[ixx'/\sin\varphi - (i/2)(x^2 + x'^2)\cot\varphi - i\varphi/2 + i\pi/4]. \qquad (15)$$

The corresponding infinitesimal operator may be obtained by evaluating for small values of φ the expression

$$\int_{-\infty}^{+\infty} f_\varphi(x, x')\psi(x')\,dx' \qquad (16)$$

(where ψ is an arbitrary function) up to terms proportional to φ. By using the method of steepest descents we obtain

$$\int_{-\infty}^{+\infty} f_\varphi(x, x')\psi(x')\,dx'$$
$$= \psi(x) + i\varphi\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 - \frac{1}{2}\right)\psi(x) + O(\varphi^2). \qquad (17)$$

Thus, the infinitesimal operator which corresponds to this infinitesimal transformation does indeed coincide up to a constant factor with the operator H_x from formulas (3). The presence of the additional term $-1/2$ is not unavoidable; its introduction does not alter the commutation relations between the operators and leads only to the ground state wavefunction remaining unchanged when the group operator acts upon it.

By utilizing formula (17) we can easily obtain the expansion of the function f_φ in terms of the oscillator eigenfunctions

$$f_\varphi(x, x') = \sum_{m=0}^{\infty} \psi_m(x)\psi_m(x')e^{im\varphi}. \qquad (18)$$

This leads directly to the relation between the function f and the Green's function for the one dimensional oscillator

*The commutation relations (19) of reference 2 contain an error: the sign of the right hand side should be reversed in formulas (8) and (11).

†To be more precise, it follows from this that there exists a one-to-one group correspondence between elements belonging to certain regions surrounding the unit elements of the two groups.

$$G(x, x', t) = \sum_{m=0}^{\infty} \psi_m(x)\psi_m(x')e^{-iE_m t} = e^{-it/2}f_{-t}(x, x')$$
$$= (2\pi \sin t)^{-1/2}\exp[-ixx'/\sin t \qquad (19)$$
$$+ (i/2)(x^2 + x'^2)\cot t - i\pi/4].$$

When $t = \pi/2 = T/4$, where $T = 2\pi$ is the period of oscillation in terms of our units, we obtain up to a constant factor the operator for the one-dimensional Fourier transformation

$$G(x, x', T/4) = (2\pi)^{-1/2}\exp(-ixx' - i\pi/4). \qquad (20)$$

Thus, if at time $t = 0$ the oscillator wavefunction in the x-representation is given by $\psi_0(x)$, while in the p-representation it is given by $\varphi_0(p)$, then at subsequent times it will vary in the following manner:

$t =$	0	$T/4$	$T/2$	$3T/4$	T
ψ:	$\psi_0(x)$	$e^{-i\pi/4}\varphi_0(x)$	$e^{-i\pi/2}\psi_0(-x)$	$e^{-i3\pi/4}\varphi_0(-x)$	$e^{-i\pi}\psi_0(x)$
φ:	$\varphi_0(p)$	$e^{-i\pi/4}\psi_0(-p)$	$e^{-i\pi/2}\varphi_0(-p)$	$e^{-i3\pi/4}\psi_0(p)$	$e^{-i\pi}\varphi_0(p)$.

Consequently the variation of the oscillator wavefunction with time may be obtained by operating on the initial wavefunction with the unitary operator of the symmetry group after setting $\varphi = -t$ in it. This unfolding in time represents a continuous transition from the coordinate wavefunction to the momentum wavefunction and conversely.

We now consider a second type of transformation in which two pairs of coordinates and momenta take part. Let the unitary matrix be of the form

$$U = \begin{pmatrix} \cos\varphi, & i\sin\varphi \\ i\sin\varphi, & \cos\varphi \end{pmatrix}. \qquad (21)$$

These matrices form a group, and for small values of φ have the form

$$U = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix} + i\varphi\begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix} + O(\varphi^2), \qquad (22)$$

i.e., the infinitesimal operator of this subgroup corresponds to the nondiagonal operator N^{ij} of the oscillator symmetry group. Transformation of coordinates in this case has the form

$$x_1' = \cos\varphi\cdot x_1 - \sin\varphi\cdot p_2, \quad x_1 = \cos\varphi\cdot x_1' + \sin\varphi\cdot p_2',$$
$$x_2' = \cos\varphi\cdot x_2 - \sin\varphi\cdot p_1, \quad x_2 = \cos\varphi\cdot x_2' + \sin\varphi\cdot p_1'. \qquad (23)$$

By constructing a system of equations analogous to system (13) and by determining the normalization coefficient in the same manner we obtain

$$f_\varphi(x_1, x_2; x_1', x_2') \qquad (24)$$
$$= (2\pi\sin\varphi)^{-1}\exp[i(x_1x_2' + x_1'x_2)/\sin\varphi - i(x_1x_2 + x_1'x_2')\cot\varphi].$$

Finally, by evaluating by the method of steepest descents the result of the action of the operator on an arbitrary function for small values of φ, we obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_\varphi(x_1, x_2; x_1', x_2')\psi(x_1', x_2')\,dx_1'dx_2' \qquad (25)$$
$$= \psi(x_1, x_2) + i\varphi(-\partial^2/\partial x_1\partial x_2 + x_1x_2)\psi(x_1, x_2) + O(\varphi^2).$$

The resultant infinitesimal operator coincides with the operators N in formulas (3).

The third type of transformation corresponding to the matrix

$$U = \begin{pmatrix} \cos\varphi, & -\sin\varphi \\ \sin\varphi, & \cos\varphi \end{pmatrix}, \qquad (26)$$

as may be easily seen, does not mix coordinates and momenta, and represents a pure rotation in the x_1, x_2 plane. In this case the infinitesimal operator will be given by the corresponding component of angular momentum.

Thus, the infinitesimal operators of the groups determined in references 1, 2, and 3 are the same. Moreover, the generating elements of the group defined in references 1 and 2 are contained in the group defined in reference 3. From this it follows that the two groups coincide.

Finally, we shall obtain up to a normalizing factor the explicit form of the kernel of the integral operator for an arbitrary matrix U. In this case the function $f(X, X')$ satisfies the system of equations

$$(\tilde{A}X' - i\tilde{B}\nabla')f = Xf, \quad (AX - iB\nabla)f = X'f, \qquad (27)$$

where obvious matrix notation has been used, while $\tilde{A}$, $\tilde{B}$ denote transposed matrices. By multiplying the two equations respectively by $\tilde{B}^{-1}$ and B^{-1} we obtain

$$\nabla' f/f = i\tilde{B}^{-1}X - i\tilde{B}^{-1}\tilde{A}X', \quad \nabla f/f = iB^{-1}X' - iB^{-1}AX. \qquad (28)$$

From the fact that the matrix U is unitary it follows directly that the matrices $\tilde{B}^{-1}\tilde{A}$, $B^{-1}A$ are symmetric and, consequently, that the equations (28) are soluble. On solving them we obtain the general form for the kernel of the integral operator

$$f = C\exp[i\tilde{X}B^{-1}X' - (i/2)(\tilde{X}'AB^{-1}X' + \tilde{X}B^{-1}AX)]. \qquad (29)$$

The coefficient C may again be determined from the requirement that the operator should leave the ground state function unchanged.

In conclusion we note that the isotropic oscillator is an example of a system in which the energy operator is completely expressed in terms of the infinitesimal operators of the symmetry group. In such cases in principle it is possible to obtain from just the symmetry properties of the system all its other properties (energy levels, degree of degeneracy, the Green's function etc.).

[1] Yu. N. Demkov, J. Exptl. Theoret. Phys. (U.S.S.R.) **26** [sic!] 757 (1954).

[2] Yu. N. Demkov, Bull. Leningrad State Univ. No. 11, 127 (1953).

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[4] S. P. Alliluev, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 200 (1957), Soviet Phys. JETP **6**, 156 (1958).

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