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CERENKOV RADIATION FROM DIPOLES

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Cerenkov radiation from electric and magnetic dipoles moving a continuous medium is considered; the radiation from dipoles moving in a channel or slit is also considered.

A particle bunch (packet) which is small compared with the wave length in the medium produce the same Cerenkov radiation as a point charge or point multiple moment corresponding to the entire bunch. Hence the Cerenkov radiation of magnetic and electric dipoles is of interest even though the dipole radiation for individual particles (neutron, electron) is very weak. Moreover, the analyses of the Cerenkov radiation from a magnetic moment given in the literature[1-6] are contradictory. For this reason we present a method of calculation which is somewhat different from that used in reference 2 to 4.

Setting $\operatorname{div}\mathbf{A} = 0$ and expanding the vector potential we obtain the relations (for simplicity it is assumed that the magnetic permeability $\mu = 1$)

$$\Delta\mathbf{A} - \frac{\varepsilon}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j} + \frac{\varepsilon}{c}\frac{\partial}{\partial t}\nabla\varphi, \qquad \Delta\varphi = -\frac{4\pi\rho}{\varepsilon},$$

$$\mathbf{A} = \sum_{\lambda,i}(q_{\lambda i}\mathbf{A}_{\lambda i} + q^*_{\lambda i}\mathbf{A}^*_{\lambda i}), \quad \mathbf{A}_{\lambda i} = c\sqrt{4\pi/\varepsilon}\,\mathbf{e}_{\lambda i}\exp\{i\mathbf{k}_\lambda\mathbf{r}\},$$

$$\mathbf{e}_{\lambda i}\cdot\mathbf{e}_{\lambda j} = \delta_{ij}, \quad \mathbf{k}_\lambda\cdot\mathbf{e}_{\lambda i} = 0, \quad i = 1,2;$$

$$\mathcal{H} = \int\frac{\varepsilon E_{tr}^2 + H^2}{8\pi}\,dV = \sum_{\lambda,i}{}' (p_{\lambda i}p^*_{\lambda i} + \omega_\lambda^2 q_{\lambda i}q^*_{\lambda i}),$$

$$\mathbf{E}_{tr} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \quad p_{\lambda i} = \frac{dq_{\lambda i}}{dt}, \quad \omega_\lambda^2 = \frac{c^2}{\varepsilon}k_\lambda^2$$

and the equation

$$\ddot{q}_{\lambda i} + \omega_\lambda^2 q_{\lambda i} = \frac{1}{c}\int(\mathbf{j}\cdot\mathbf{A}_{\lambda j})\,dV$$

$$= \sqrt{4\pi/\varepsilon}\,\{e(\mathbf{e}_{\lambda i}\cdot\mathbf{v}) + ic\mathbf{m}\cdot[\mathbf{k}_\lambda\times\mathbf{e}_{\lambda i}] \tag{1}$$

$$+ i(\mathbf{e}_{\lambda i}\cdot\mathbf{p})(\mathbf{k}_\lambda\cdot\mathbf{v})\}\exp\{i\mathbf{k}_\lambda\times\mathbf{v}t\},$$

where

$$\mathbf{j} = \rho\mathbf{v} + c\,\mathbf{curl}\,\mathbf{M} + \frac{\partial\mathbf{P}}{\partial t} \tag{2}$$

$$= e\mathbf{v}\delta(\mathbf{r} - \mathbf{v}t) + c\,\mathbf{curl}([\mathbf{m}\delta(\mathbf{r} - \mathbf{v}t)] + \frac{\partial}{\partial t}[\mathbf{p}\delta(\mathbf{r} - \mathbf{v}t)].$$

Integrating Eq. (1), for example with the initial conditions $q_{\lambda i}(0) = P_{\lambda i}(0) = 0$ we find the energy $\mathcal{H}$. Introducing the density of states $dZ_i(\omega) = (2\pi c)^{-3}\epsilon^{3/2}\omega^2\,d\omega\,d\Omega$, isolating the resonance term in $\mathcal{H}$, which increases in proportion to time, and integrating over the polar angle θ, we obtain an expression for the energy radiated per unit time:

$$\mathcal{H} = \frac{1}{2\pi vc^2}\sum_{i=1,2}\int d\omega\int_0^{2\pi} n^2\omega^3\left\{\mathbf{m}[\mathbf{k}_1\times\mathbf{e}_i] + \frac{1}{n}(\mathbf{e}_i\cdot\mathbf{p})\right\}^2 d\varphi, \tag{3}$$

where $n^2(\omega) = \epsilon(\omega)$ $\cos\theta = c/vn(\omega)$, $\mathbf{k}_1 = \mathbf{k}/k$, θ and φ are the polar and azimuthal angles in a coordinate system in which the z axis is in the direction of the velocity $\mathbf{v}$. In Eq. (3) it is assumed that the charge $e = 0$: if $e \neq 0$, the radiation is equal to the sum of Eq. (3) and the well-known expression for the Cerenkov radiation of a charge. The integration over frequency in Eq. (3) is carried out over the frequency region for which $c/vn(\omega) \leq 1$. At first glance, it may seem that in this calculation no account has been taken of the dispersion in n. However, one is easily convinced that this is not the case and that dispersion has properly been introduced in Eq. (3).

If the magnetic moment (also the electric dipole) is in the direction of $\mathbf{v}$, we obtain from Eq. (3) the standard expression.[1-3] If there is only a magnetic moment $\mathbf{m}_0$ in the system in which the particles (bunch) are at rest, the moments perpendicular to the velocity in Eq. (3) must be taken as $\mathbf{m} = \mathbf{m}_0$ and $\mathbf{p} = \mathbf{v}\times\mathbf{m}/c$. In this case we obtain

$$\mathcal{H} = \frac{m^2}{2vc^2}\int n^2\omega^3\left\{2\left(1 - \frac{1}{n^2}\right)^2 - \left(1 - \frac{v^2}{c^2n^2}\right)\left(1 - \frac{c^2}{v^2n^2}\right)\right\}d\omega. \tag{4}$$

This expression coincides with that given by Frank in reference 2; however, it differs from those given in references 3 and 6. Thus, in reference 3 the following expression appears in place of Eq. (4)

$$\mathcal{H} = \frac{m^2v}{2c^4}\int n^1\omega^3\left(1 - \frac{c^2}{v^2n^2}\right)^2 d\omega. \tag{5}$$

The reason for the discrepancy lies in the fact that in references 3 and 6 use was made of magnetic dipoles formed by magnetic poles. When $\epsilon \neq 1$, such moving "true" magnetic dipoles are not the same as moving current-moments.* More specifically, in place of the moment $(\mathbf{v}\times\mathbf{m})/c$ we have[3] the electric moment $\epsilon(\mathbf{v}\times\mathbf{m})/c$; this is equivalent to taking account of the electrical polarization of the medium due to the dipole itself.[4] It is curious that this case can be realized for bunches; it is necessary (for the frequency being considered) that ϵ, the dielectric permittivity in the bunch itself, be the same as ϵ, that of the surrounding medium (for example, a plasma in a magnetic field).

Using a quantum-mechanical calculation,[1,7-9] starting from the Pauli equation or the Dirac equation (if we do not consider the radiation from the charge) we obtain expressions such as Eq. (4) and similar formulas for a moment parallel to $\mathbf{v}$ (cf. reference 1). In this case, if the spin in the initial state is parallel or anti-parallel to the velocity $\mathbf{v}$, an expression such as Eq. (4) is obtained only in transitions in which there is spin flip since it is only in this case that there are components of the spin operator which are perpendicular to $\mathbf{v}$. Consequently, the Cerenkov radiation of the magnetic moment of the electron or neutron has essentially no quantum-mechanical attributes.

A characteristic feature of Eq. (4) is the fact that the integrand does not vanish at threshold (for $\cos\theta = c/nv = 1$):

$$\mathscr{H} = \frac{m^2}{vc^2}\int n^2\omega^3\left(1-\frac{1}{n^2}\right)^2 d\omega$$

(We may note that Loskutov and Kukanov[9] have obtained precisely the same expression from a quantum-mechanical calculation). This result, however, need not be considered paradoxical since the total energy $\mathcal{K}$ vanishes at threshold and then increases smoothly. Actually, if dispersion is taken into account, as the velocity increases the radiation is observed only at frequencies which correspond to the maximum value of $n(\omega)$. Furthermore, if recoil is taken into account, as is automatically the case in the quantum-mechanical calculation,[1]

$$\cos\theta = \frac{c}{nv} + \frac{\hbar\omega\,(c^2/v^2-1)^{1/2}(n^2-1)}{2mc^2 n}, \qquad (6)$$

where m is the rest mass of the particle or bunch. By virtue of Eq. (6), even if $n = \text{const}$ the radiation starts at one frequency as v increases; in the present case this is the frequency $\omega = 0$. Thus, as v increases there is a gradual expansion of the region of integration and the value of (4) increases.

We may note that an expression such as Eq. (5) can be obtained from a quantum-mechanical calculation. For this purpose, in the Dirac equation for a charged particle we add an appropriate term, proportional to $\gamma_i\gamma_k G_{ik}$; for a particle with non-kinematic magnetic moment, we replace $\gamma_i\gamma_k F_{ik}$ by $\gamma_i\gamma_k H_{ik}$ (here $F_{ik} = \{\mathbf{H}, i\mathbf{E}\}$, $H_{ik} = \{\mathbf{H}, i\mathbf{D}\}$ $\mathbf{D} = \epsilon\mathbf{E}$, $G_{ik} = F_{ik} - H_{ik}$). There is no basis for introducing these changes in the application to an individual particle; when the quantum-mechanical calculation is applied to bunches, however, there is no objection.

In conclusion we consider the Cerenkov radiation due to the motion of dipole moments in voids — channels or slits (for simplicity we assume $\epsilon = 1$ and $\mu = 1$ in the void). In the case of a charged particle it is well known[10,11] that as the radius of the channel or the width of the slit approaches zero the Cerenkov radiation becomes the same as that characteristic of motion in a continuous medium (when $\cos\theta \sim 1$ this occurs when $a/\lambda \ll 1$, where a is the radius of the channel or the width of the slit and $\lambda = \lambda_0/n$ is the wave length in the medium). At first glance it might seem that this result should also apply for dipoles and other multipoles; in general, however, this is not the case.

To compute the effect of a thin channel (slit) on the Cerenkov radiation it is convenient to make use of the reciprocity theorem

$$\int_{(1)} \mathbf{j}_\omega^{(1)}\mathbf{E}_\omega^{(2)}dV = \int_{(2)} \mathbf{j}_\omega^{(2)}\mathbf{E}_\omega^{(1)}dV,$$

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*If we introduce magnetic poles with density $\rho_m(\mathbf{r})$ the field equation have the form (we take $\rho = 0$, $\mathbf{j} = 0$ and $\mathbf{B} = \mu\mathbf{H}$).

$$\text{curl}\,\mathbf{H} = \frac{1}{c}\frac{\partial\varepsilon\mathbf{E}}{\partial t}, \qquad \text{div}\,\varepsilon\mathbf{E} = 0,$$

$$\text{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial\mu\mathbf{H}}{\partial t} - \frac{4\pi}{c}\rho_m\mathbf{v}, \qquad \text{div}\,\mu\mathbf{H} = 4\pi\rho_m,$$

whence

$$\text{curl curl}\,\mathbf{H} + \frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = -\frac{4\pi}{c}\varepsilon\frac{\partial(\rho_m\mathbf{v})}{\partial t},$$

$$\text{curl curl}\,\mathbf{E} + \frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = -\frac{4\pi}{c}\,\text{curl}(\rho_m\mathbf{v}).$$

However, if there are electric charges and currents

$$\text{curl curl}\,\mathbf{H} + \frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = \frac{4\pi}{c}\,\text{curl}(\rho\mathbf{v}),$$

$$\text{curl curl}\,\mathbf{E} + \frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = -\frac{4\pi}{c}\mu\frac{\partial(\rho\mathbf{v})}{\partial t}.$$

In other words, the equations for magnetic poles are obtained from the charge equations by making the substitutions

$$\mathbf{E}\to\mathbf{H}, \qquad \mathbf{H}\to-\mathbf{E}, \qquad \rho\to\rho_m, \qquad \mu\to\varepsilon.$$

Thus, when $\mu = 1$ the magnetic moment due to a current is completely equivalent to the "true" magnetic dipole only in vacuum, where $\varepsilon = 1$.

where the $\mathbf{j}_\omega^{(1,2)}$ are the Fourier components of the "transverse" current density in regions 1 and 2; the field $\mathbf{E}_\omega^{(2)}$ is produced by current 2 in region 1 and field $\mathbf{E}_\omega^{(1)}$ is produced by current 1 in region 2. Writing the current in the form $\mathbf{j} = \rho\mathbf{v} + \partial\mathbf{P}/\partial t + c \operatorname{curl} \mathbf{M}$ we have:

$$\int\limits_{(1)} [(\rho\mathbf{v})_\omega^{(1)} \cdot \mathbf{E}^{(2)} + i\omega\mathbf{P}_\omega^{(1)} \cdot \mathbf{E}_\omega^{(2)} - \mu\mathbf{M}_\omega^{(1)} \cdot \mathbf{H}_\omega^{(2)}]\, dV$$
$$= \int\limits_{(2)} [(\rho\mathbf{v})_\omega^{(2)} \cdot \mathbf{E}_\omega^{(1)} + i\omega\mathbf{P}_\omega^{(2)} \cdot \mathbf{E}_\omega^{(1)} - \mu\mathbf{M}_\omega^{(2)} \cdot \mathbf{H}_\omega^{(1)}]\, dV. \tag{7}$$

For the Cerenkov radiation of a point charge moving along the z axis we have

$$(\rho\mathbf{v})_\omega^{(1)} = \frac{e}{2\pi}\mathbf{v} e^{-i\omega z/v} \delta(x)\,\delta(y),$$

and placing an electric dipole $\mathbf{p}^{(2)}$ at point 2, removed from the trajectory, we have

$$\frac{e}{2\pi}\int \mathbf{v}\cdot\mathbf{E}^{(2)}(0,0,z)\, e^{-i\omega z/v}\, dz = i\omega\mathbf{p}^{(2)}\cdot\mathbf{E}\,(2), \tag{8}$$

where $\mathbf{E}(2) \equiv \mathbf{E}^{(1)}(2)$ is the radiation field at point 2. If the charge moves in a thin channel or narrow slit (i.e., if $a/\lambda \ll 1$) the quantity $\mathbf{v}\cdot\mathbf{E}^{(2)}(0, 0, z)$ remains the same as for a continuous medium since the tangential component of $\mathbf{E}^{(2)}$ is continuous. Hence, as is apparent from Eq. (8) the radiation field $\mathbf{E}$ also remains the same as in a continuous medium. For the radiation of an electric dipole, where $\mathbf{P}^{(1)} = \mathbf{p}\,\delta(z - vt)\,\delta(x)\,\delta(y)$,

$$\int \mathbf{p}\cdot\mathbf{E}^{(2)}(0,0,z) e^{-i\omega z/v}\, dz = \mathbf{p}^{(2)}\cdot\mathbf{E}\,(2). \tag{9}$$

If the dipole $\mathbf{p}$ is parallel to the axis of the channel or lies in the plane of the slit when $a/\lambda \ll 1$ the radiation field is once again the same as for a continuous medium. For a dipole which is perpendicular to the plane of the slit

$$\mathbf{p}\cdot\mathbf{E}^{(2)}(0,0,z) = \varepsilon(\omega)\,\mathbf{p}\cdot\mathbf{E}_0^{(2)}(0, 0, z),$$

where $\mathbf{E}_0^{(2)}$ is the field in the continuous medium produced by dipole 2. If we express the Cerenkov field of dipole 1 (with moment $\mathbf{p}$) in the continuous medium in terms of $\mathbf{E}_0$,

$$\int \mathbf{p}\cdot\mathbf{E}_0^{(2)} e^{-i\omega z/v}\, dz = \mathbf{p}^{(2)}\cdot\mathbf{E}_0(2),$$

$$\int \mathbf{p}\cdot\mathbf{E}^{(2)} e^{-i\omega z/v}\, dz = \varepsilon\int \mathbf{p}\cdot\mathbf{E}_0^{(2)} e^{-i\omega z/v}\, dz = \mathbf{p}^{(2)}\cdot\mathbf{E}\,(2).$$

Whence $\mathbf{E} = \epsilon\mathbf{E}_0$. In the case where the dipole is perpendicular to the axis of a thin channel which is in the form of a circular cylinder $\mathbf{E} = \mathbf{E}_0 \cdot 2\epsilon\,(\epsilon + 1)$. Since the magnetic wave in the field zone is proportional to the electric field the energy radiated in the case of a slit or a channel is increased respectively by a factor of ϵ^2 or $[2\epsilon/(\epsilon+1)]^2$.* A dipole with arbitrary orientation can be analyzed in terms of dipoles that are perpendicular and parallel to the axis of the channel (slit); thus, using the principle of superposition the problem is reduced to one of the foregoing. As is apparent from Eq. (7), the presence of the channel has no effect on the radiation of a magnetic dipole $\mathbf{m}$ when $\mu = 1$. If there are both electric and magnetic dipoles the radiated fields are combined (but obviously not the energy); thus the problem can again be solved.

Obviously a moving current moment and a "true" magnetic moment placed in a void must give the same radiation. This conclusion was verified by Bogdankevich[12] by a direct calculation for different dipoles moving in a circular channel; in particular, in the case of a thin channel it is found, as is to be expected, that the result is the same as the derived result — the field of the electric dipole is increased by a factor of $2\epsilon/(\epsilon+1)$.

In connection with the fact that the Cerenkov radiation of a moving dipole (with $\mu \neq 1$ also a magnetic dipole) depends on the shape of a cavity which can be as narrow as desired, there is some question as to the validity of Eqs. (3) and (4) for the motion of a dipole in a continuous medium. It is clear from the reciprocity theorem that here we are examining the validity of assuming that the field which acts on the dipole $\mathbf{E}_d$ is the average macroscopic field $\mathbf{E}$. If a fixed dipole is introduced into a medium, in general this is not the case (i.e., $\mathbf{E}_d \neq \mathbf{E}$). However, if we consider a particle with charge or a dipole moment which moves along a given trajectory, the average field which acts on a "physically infinitesimally small" portion of the path is precisely the macroscopic field. This same conclusion applies in connection with the original expression (1) for the motion of a particle in a continuous medium; this expression (or the equivalent wave equation) is obtained by averaging the equations of microscopic electrodynamics. Thus, in our opinion the validity of Eqs. (3) and (4) for Cerenkov radiation of point dipoles in a continuous medium should raise no objection.

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*To compute the radiated energy from Eq. (3) for the case of a channel or slit, we must replace $\mathbf{p}$ by the appropriate expression determined from Eq. (9); for example, in the case of a dipole perpendicular to the axis of a circular channel $\mathbf{p}$ is replaced by $[2\varepsilon/(\varepsilon + 1)]\mathbf{p}$.

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