

EXCHANGE EFFECTS IN STRIPPING REACTIONS

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Formulas for the differential cross sections of the stripping reactions (d, p) and (d, n) have been obtained in the Born approximation, with inclusion of effects of the antisymmetry of the total wave function. The most important case, that of the shell configurations  $j^{n-1}$  for the initial nucleus and  $j^n$  for the final nucleus, has been considered.

1. GENERAL THEORY

It has been shown in a number of papers that in particular cases of calculations of stripping reaction cross-sections the antisymmetrization of the total wave function can affect the results of the calculations to an important extent.<sup>1</sup> Calculations on stripping reactions with inclusion of the antisymmetrization have not been carried out, however, unless one counts the work of French, which is of a preliminary nature.

We shall use the results of the general theory of scattering.<sup>2,3</sup> We first carry through a treatment without antisymmetrization. We assign the numbers 0 and 1 to the proton and neutron in the deuteron;  $V(01)$  is the interaction between them. The interaction of the proton with the other nucleons of the nucleus is described by an averaged potential  $V(0\xi)$ , taken, for example, from the optical model. Then the wave function is determined from the equation

$$\Psi^{(+)} = \Phi + (1/a^{(\pm)})V(01)\Phi, \quad a^{(\pm)} = E \pm is - H, \quad (1)$$

where  $H$  is the total Hamiltonian of the system, and  $\Phi$  is the wave function of the initial state [reaction (p, d)].

The antisymmetrization is performed by the application of an operator  $A$  (whose concrete form will be given below):

$$\bar{\Psi}^{(+)} = A\Psi^{(+)} = A\Phi + (1/a^{(\pm)})AV\Phi. \quad (2)$$

To obtain the expression for the reaction amplitude in the Born approximation, we replace  $1/a$  by  $1/a'(01)$ , using the fact<sup>3</sup>

$$1/a = \{1/a'(01)\} [1 + \{H'(01)/a\}],$$

where  $a'(01)$  is the Green's function of the final system,<sup>2</sup> and  $H_{fin} = H - H'(01)$  is its Hamiltonian.

The reaction amplitude is given by the second term in Eq. (2), that is,

$$AV\Phi/a'(01). \quad (3)$$

Thus our task reduces to the calculation, in Born approximation with distorted waves for the protons and deuterons, of the matrix element

$$\int \Psi_{fin}^*(2, 3, \dots, n) \Psi_{kd}^*(0, 1) AV \Psi_{int}(1, 2 \dots n) \times \Psi_{kp}(0) dr_0 \dots dr_n, \quad (4)$$

where  $\Psi_{fin}$  is the wave function of the final nucleus,  $\Psi_{init}$  is that of the initial nucleus,  $\Psi_{Kp}$  is the wave function of the proton in the field of the initial nucleus, and  $\Psi_{Kd}$  is that of the deuteron. The antisymmetrization operator  $A$  is defined in the following way:

$$AV\Psi_{int}(1, 2 \dots n) \Psi_{kp}(0) = \frac{1}{\sqrt{n+1}} \sum_{i+j} (-1)^{n+1-i} V(i, j) \times \Psi_{int}(0, 1 \dots i-1, i+1, \dots n) \Psi_{kp}(i). \quad (5)$$

The normalization factor in this case is found from the requirement of conservation of flux. The antisymmetrization procedure used in Eq. (5) was indicated by Racah in reference 4, where the antisymmetrization was carried out for the wave function of the shell configuration  $l^n l'$ .

2. CALCULATION WITH WAVE FUNCTIONS OF THE SHELL THEORY

We shall suppose that

$$\Psi_{int} = |j^n \alpha_1\rangle, \quad \Psi_{fin} = |j^{n-1} \alpha_2\rangle,$$

where  $\alpha$  is the set of quantum numbers necessary for unique specification of a state. Using Racah's technique,<sup>4,5</sup> with the expression (5) for the operator  $A$ , we work out the matrix element (4):

$$\begin{aligned}
I &= 2I_1 + (n-1)2I_2 + 2(n-1)(n-2)I_3; \\
I_1 &\sim \langle k_d S_d(12) | V(12) | j(1) k_p S_p(2) \rangle; \\
I_2 &\sim \sum_{\alpha_1, \alpha_2} \langle k_d S_d(12), j(3) | V(13) | j^2 \alpha_4(12) k_p S_p(3) \rangle \\
&\quad \times \langle j^n \alpha_1 | j^{n-2} \alpha_3; j^2 \alpha_4 \rangle \langle j^{n-1} \alpha_2 | j^{n-2} \alpha_3 \rangle; \\
I_3 &\sim \sum_{\alpha_1, \alpha_2, \alpha_3} \langle j^2 \alpha_8(12) | V(12) | j(1) k_p S_p(2) \rangle \langle k_d S_d(34) | j^2 \alpha_6(34) \rangle \\
&\quad \times \langle j^n \alpha_1 | j^{n-2} \alpha_5; j^2 \alpha_6 \rangle \langle j^{n-1} \alpha_2 | j^{n-3} \alpha_7; j^2 \alpha_8 \rangle \langle j^{n-2} \alpha_5 | j^{n-3} \alpha_7 \rangle.
\end{aligned} \tag{6}$$

Here  $\langle j^n | j^{n-1} \rangle$  and so on are fractional parentage coefficients. Thus the calculation of the differential cross-section reduces to the calculation of  $I_1^2, I_2^2, I_1 I_2$ , etc.

In carrying out the sums of products of Clebsch-Gordan coefficients over magnetic quantum numbers we have used the effective graphic method of Levinson.<sup>6</sup> We thus arrive at the following formulas (the expressions for the interference terms are not presented here):

$$\begin{aligned}
I_2^2 &= (4\pi)^2 [J_1] [J_2] [j]^3 [l] (C_{M_{\tau_1}^{\tau_1} \tau_2}^{T_1 \tau_1 T_2})^2 \sum [a] [b] \begin{Bmatrix} l & s & j \\ l & s & j \\ L_4 & S_4 & J_4 \end{Bmatrix} \sqrt{[S_4] [J_4] [L_4] [L_0]} \langle j^n \alpha_1 | j^{n-2} \alpha_3; j^2 \alpha_4 \rangle \langle j^{n-1} \alpha_2 | j^{n-2} \alpha_3 \rangle \hat{C}_{000}^{LL'a} \\
&\quad \times \hat{C}_{000}^{L_2 L_4 a} \hat{C}_{000}^{LL_0} \hat{C}_{000}^{l k L_0} \hat{C}_{000}^{k l_2 l} \hat{C}_{000}^{L_4 l} E_{k_d k_p}(l, k, l_2, L) \sqrt{[S'_4] [J'_4] [L'_4] [L'_0]} \begin{Bmatrix} l & s & j \\ l & s & j \\ L'_4 & S'_4 & J'_4 \end{Bmatrix} \langle j^n \alpha_1 | j^{n-2} \alpha'_3; j^2 \alpha'_4 \rangle \\
&\quad \times \langle j^{n-1} \alpha_2 | j^{n-2} \alpha'_3 \rangle \hat{C}_{000}^{L' k' L'_0} \hat{C}_{000}^{l k' L'_0} \hat{C}_{000}^{k' l'_2 l'} \begin{Bmatrix} L'_4 & l & l \\ k' & L'_0 & l'_2 \end{Bmatrix} E_{k_d k_p}^*(l, k', l'_2, L') (-1)^{p_{22}} \begin{Bmatrix} L'_0 & L' & a & l'_2 \\ l & L & l_2 & L_4 \\ b & l & L_0 & L_4 \end{Bmatrix} \begin{Bmatrix} J_4 & L_4 & S_d \\ L'_4 & J'_4 & b \end{Bmatrix} \\
&\quad \times \begin{Bmatrix} J_3 & J_4 & J_1 \\ J'_4 & J'_3 & b \end{Bmatrix} \begin{Bmatrix} J_3 & j & J_2 \\ j & J'_3 & b \end{Bmatrix} \begin{Bmatrix} l & j & s \\ j & l & b \end{Bmatrix} P_a(\cos \theta).
\end{aligned} \tag{8}$$

Here  $S_d$  is the spin of the deuteron,  $s$  that of the nucleon;  $\rho_{22} = J_1 + J_2 + l + s + l_2 + L + a$ ; the summation is taken over all indices except  $l, s, j$ ,

$S_d, \alpha_1, \alpha_2$  [ $\alpha = (J, T, \text{seniority})$ ]. We also make use of the facts that  $S_4 = S'_4 = S_d = 1, T_4 = T'_4 = T_d = 0, J_4 + T_4$  is even,  $L_4 + S_4 + T_4$  is odd;

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = (-1)^{j_1+j_2+j_3+j_4} W(j_1 j_2 j_3 j_4; j_5 j_6), \quad W \text{ is a Racah coefficient; } \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{Bmatrix} \text{ is a } 9j \text{ symbol}^{7,8};$$

$$\begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{Bmatrix} \text{ is a } 12j \text{ symbol of the second kind}^8; \quad \begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{Bmatrix} = \sum_x (-1)^l [x] \begin{Bmatrix} j_1 & k_1 & x \\ k_2 & j_2 & l_1 \end{Bmatrix} \begin{Bmatrix} j_2 & k_2 & x \\ k_3 & j_3 & l_2 \end{Bmatrix} \begin{Bmatrix} j_3 & k_3 & x \\ k_4 & j_4 & l_3 \end{Bmatrix} \begin{Bmatrix} j_4 & k_4 & x \\ k_1 & j_1 & l_4 \end{Bmatrix},$$

$f$  is the sum of all the parameters of the 12j symbol;

$$\hat{C}_{000}^{l_1 l_2 l_3} = \sqrt{[l_1] [l_2] / 4\pi [l_3]} C_{000}^{l_1 l_2 l_3}; \quad \theta \text{ is the angle between } k_d \text{ and } 1/2 k_d - k_p;$$

$$E_{k_d, k_p}(l, k, l_2, L) = V_0 i^{l_2 - L} \int_R^\infty f_k(r_1, r_2) j_{l_2} \left( \frac{1}{2} k_d r_2 \right) j_L \left( \left| \frac{1}{2} k_d - k_p \right| r_1 \right) \varphi_l^2(r_1) \varphi_l(r_2) r_1^2 r_2^2 dr_1 dr_2,$$

where  $\varphi_l(r)$  is the radial part of the wave number of a bound nucleon with the orbital angular momen-

tum  $l$ ;  $f_k(r_1, r_2)$  is defined in terms of the internal wave function of the deuteron

$$\chi_d(\mathbf{r}_1 - \mathbf{r}_2) = \sum_k j_k(r_1, r_2) |k\rangle P_k(\cos \omega_{12}),$$

in particular<sup>1,9</sup>

$$e^{-\alpha r_{12}} / r_{12} = \alpha \sum_k [k] j_k(i\alpha r_1) h_k^{(1)}(i\alpha r_2) P_k(\cos \omega_{12})$$

for  $r_1 < r_2$ ;  $V_0 = 4\pi\hbar^2 M\alpha$  [the interaction potential of the nucleons is taken in the form  $V_0\delta(\mathbf{r}_1 - \mathbf{r}_2)$ ].

$$\begin{aligned} I_3^2 &= \frac{1}{4\pi} (C_{M_{T_1}^{T_1} T_2}^{T_1 T_2})^2 [J_1][J_2][j][T_1] \sum C_{j^{n-2\alpha_i}; i^{\alpha_i}}^{j^{\alpha_i}} \\ &\times C_{j^{n-3\alpha_i}; i^{\alpha_i}}^{j^{n-2\alpha_i}} C_{j^{n-3\alpha_i}; \tau}^{j^{n-2\alpha_i}} \left\{ \begin{matrix} T_2 & T_1 & \tau \\ \tau & T_5 & T_7 \end{matrix} \right\} \left\{ \begin{matrix} T_2 & T_1 & \tau \\ \tau & T_5' & T_7' \end{matrix} \right\} [T_5]^{1/2} [T_5']^{1/2} \\ &\times (-1)^{2T_1+1+T_7+T_7'} \hat{C}_{000}^{l L_s} \hat{C}_{000}^{L_s l} \hat{C}_{000}^{l' l'} \hat{C}_{000}^{L_s' l' l'} \\ &\times \hat{C}_{000}^{l' p} \hat{C}_{000}^{L_s' l' p} \sqrt{[J_6][J_6'] [J_8][J_8'] [J_5][J_5'] [S_5][S_5'] [l_p][l_p']} \\ &\times (-1)^{2s} \left\{ \begin{matrix} L_8 & J_8 & S_8 \\ J_8' & L_s' & a \end{matrix} \right\} \left\{ \begin{matrix} J_8 & J_6 & J_1 \\ J_6' & J_8' & a \end{matrix} \right\} \left\{ \begin{matrix} J_6' & J_2 & b \\ J_5' & j & J_7' \end{matrix} \right\} \left\{ \begin{matrix} J_6 & J_2 & c \\ J_5 & j & J_7 \end{matrix} \right\} \left\{ \begin{matrix} j & l & s \\ J_5' & L_s' & S_5' \\ b & l_p' & s \end{matrix} \right\} \\ &\times \left\{ \begin{matrix} j & l & s \\ J_5 & L_s & S_5 \end{matrix} \right\} \left\{ \begin{matrix} J_6 & c & J_2 \\ b & J_6' & a \end{matrix} \right\} \left\{ \begin{matrix} c & l_p & s \\ l_p' & b & a \end{matrix} \right\} P_a(\cos \gamma) A_{k_d}(l^2, L_8) \\ &\times A_{k_d}^*(l^2, L_8') B_{k_p}(l, l_p) B_{k_p}^*(l, l_p') + \text{comp. conj.} \end{aligned} \quad (9)$$

where

$$\begin{aligned} A_{k_d}(l^2, L_8) &= 4\pi i^{-L_8} \int_R^\infty j_{L_8}(k_d R) Y_{L_8, M_8}^*(\mathbf{R}_d) \chi_d(\mathbf{r}) \\ &\times \Psi_{L_8, M_8}(\mathbf{r}_1 \mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2, \quad \mathbf{R}_d = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2; \\ B_{k_p}(l, l_p) &= 4\pi i^{l_p} V_0 \int_R^\infty \varphi_l^2(r) j_{l_p}(k_p r) r^2 dr; \end{aligned}$$

$\rho_{33} = J_1 + J_2 - s + 1 + b + c + J_7 + J_7'$ ; in the summation we use the facts that  $S_8 = S_d = 1$ ,  $T_8 = T_d = 0$ ,  $J_8 + T_8$  is odd,  $L_8 + S_8 + T_8$  is odd;  $\gamma$  is the angle between  $\mathbf{k}_p$  and  $\mathbf{k}_d$ .

$$\frac{d\sigma}{d\Omega} = \frac{M_p M_d}{4\pi^2 \hbar^4} \frac{k_p}{k_d} \frac{1}{[J_2][S_d]} \frac{n}{2} I^2, \quad (10)$$

for the reaction  $A(d, p)A + 1$ .

The calculation of the radial integrals B, D, and E is simple in principle. We can make an estimate of the integrals A if we extend the integration over the entire region and use oscillator wave functions for the bound nucleons, which makes it possible to use a simple transformation to separate the motion of the center of mass of two nucleons from their relative motion.<sup>10</sup> For the integrals B, D, and E it is natural to use as the wave functions of the bound nucleon the functions  $h_l^{(1)}(\kappa r)$ . They are normalized in the following way:<sup>11</sup>  $[\varphi_l(R)/h_l^{(1)}(\kappa R)] h_l^{(1)}(\kappa r)$ ;  $\varphi_l(R)$  is the amplitude of the single-particle wave function at

the surface of the nucleus. The Wigner limit for the reduced width corresponds to the value of  $\varphi_l(R)$  given by:<sup>12</sup>  $R^3 [\varphi_l(R)]^2 / 2 = 1$ .

The usual theory of stripping involves only one radial integral  $D_{\mathbf{k}_d, \mathbf{k}_p}(l)$ , and its absolute value

does not affect the angular distribution. Use of plane waves in the crudest sort of calculation gives for the particles topping the barrier an increase of the radial integral by a factor of five to seven, in the case of reactions with light nuclei.<sup>11,13</sup> The facts show,<sup>14</sup> however, that the relative values of the absolute cross-sections for cases of capture of nucleons with different values of  $l$  are well reproduced by the expression  $D^2(l)$ .

Our results involve a large number of radial integrals, so that the crudeness with which they are estimated makes the results, generally speaking, rather indefinite. Nevertheless the fact noted above allows us to hope that for each series of integrals the dependences on  $k$ ,  $L$ ,  $l_p$ , and so on are given correctly, so that in the calculation of angular distributions we can try to confine ourselves to the two unknown constant coefficients of  $I_2$  and  $I_3$ , with a crude preliminary estimate of their order of magnitude.

The calculation of the expressions containing spherical functions can be carried out more simply by directing the  $z$  axis along one of the momenta, for example  $\frac{1}{2}\mathbf{k}_d - \mathbf{k}_p$ , or  $\mathbf{k}_d$ , etc., depending on the particular term.

### 3. CONCLUDING REMARKS

The cases that are the simplest and most interesting to analyze occur when one or two of the three terms in the expression (6) for the matrix element vanish. If  $n = 1$ , there remains only  $I_1$ , and we arrive at the usual formulas of the theory of stripping. If  $n = 2$ , the term in  $I_3$  vanishes. If we consider the inverse process, the proton is incident on an odd-odd nucleus with two nucleons outside a closed shell. The structure of the expression (6) for the amplitude  $I_2$  shows that  $I_2$  is the reaction amplitude for the process of "knocking out" of a deuteron from the nucleus by the proton, which is itself bound in the final state.  $I_3^2$  corresponds to the process of "heavy particle stripping,"<sup>15,16</sup> although it is not a complete expression for the squared amplitude of this process. In this case the angular distribution will have a maximum in the backward hemisphere.  $I_2^2$  will evidently not have this peculiarity, since the "knocking out" of a particle is as a rule associated with a maximum in the forward part of the angular distribution. This argument agrees qualitatively with the angular dis-

tribution<sup>17</sup> of the protons from the reaction  $B^{10}(dp)B^{11*}(J^* = \frac{1}{2}^-)$ , which is mainly due to a term of the type  $I_2^2$ . In fact, in a case in which  $J_{fin} = J_1$  cannot be obtained by vector composition of  $J_{init} = J_2$  and  $j$ ,  $I_1 = 0$ , i.e., ordinary "stripping" is "forbidden" in the shell theory. There remain the terms in  $I_2$  and  $I_3$ . An example of this latter type occurs in the reaction  $B^{10}(n,d)Be^{9*}$  with formation of the  $Be^9$  in the first excited state  $J^* = \frac{1}{2}^-$ .

For the calculation of the cross-section of the reaction  $B^{10}(d,p)B^{11*}$  mentioned above, the final formulas must be changed somewhat, since the final nucleus is described by the configuration  $j^6j'$ ,  $j = \frac{3}{2}^-$ ,  $j' = \frac{1}{2}^-$ .

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<sup>1</sup>A. P. French, Phys. Rev. **107**, 1655 (1957).  
Kammuri, Nakasima, and Takagi, Prog. Theor. Phys. **17**, 306 (1957).

<sup>2</sup>B. A. Lippmann, Phys. Rev. **102**, 264 (1956).

<sup>3</sup>G. Takeda and K. M. Watson, Phys. Rev. **97**, 1336 (1955).

<sup>4</sup>G. Racah, Phys. Rev. **63**, 367 (1943).

<sup>5</sup>G. Racah, Phys. Rev. **62**, 438 (1942). A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. **A214**, 515 (1952).

<sup>6</sup>I. B. Levinson, Тр. Физ.-техн. ин-та АН ЛитССР (Trans. Phys.-Tech. Inst. Acad. Sci. Lithuanian S.S.R.) **2**, 17, 31 (1957); Тр. АН ЛитССР (Trans. Acad. Sci. Lithuanian S.S.R.) **B4**, 3 (1957).

<sup>7</sup>A. de-Shalit, Phys. Rev. **91**, 1479 (1953). Arima, Horie and Tanabe, Prog. Theor. Phys. **11**, 143 (1954). H. Jahn and J. Hope, Phys. Rev. **93**, 318 (1954).

<sup>8</sup>I. B. Levinson and V. V. Vanagas, Оптика и спектроскопия (Optics and Spectroscopy) **2**, 10 (1957).

<sup>9</sup>W. Swiatecki, Proc. Roy. Soc. **A205**, 238 (1951).

<sup>10</sup>I. Talmi, Helv. Phys. Acta **25**, 185 (1952).

<sup>11</sup>W. Tobocman and M. H. Kalos, Phys. Rev. **97**, 132 (1955).

<sup>12</sup>A. M. Lane, Proc. Phys. Soc. **A66**, 977 (1953).

<sup>13</sup>K. Standing, Phys. Rev. **101**, 152 (1956).

<sup>14</sup>J. Holt and T. Marsham, Proc. Phys. Soc. **A66**, 1032 (1953).

<sup>15</sup>T. Fulton and G. E. Owen, Phys. Rev. **108**, 789 (1957).

<sup>16</sup>L. Madansky and G. E. Owen, Phys. Rev. **99**, 1608 (1955).

<sup>17</sup>N. Evans and W. Parkinson, Proc. Phys. Soc. **A67**, 684 (1954).

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**ERRATA TO VOLUME 7**

Page	Reads	Should Read
533, title	Nuclear magnetic moments of Sr <sup>87</sup> and Mg <sup>95</sup>	Nuclear magnetic moments of Sr <sup>87</sup>
645 Eq. (1)	$\dots + \alpha \sqrt{j_0(j_0 + 1)}$	$\dots - \alpha \sqrt{j_0(j_0 + 1)}$
647 Eq. (11)	$(L + 1)  B_L^- ^2 - L  B_L^+ ^2$	$L(L + 1) [  B_L^- ^2 -  B_L^+ ^2 ]$
894 Eq. (12)	$\epsilon_{11} = 1 - \sum \frac{\dots}{\sqrt{\pi/\mu}}$	$\epsilon_{11} = 1 - \sum \frac{\dots}{\sqrt{\pi \mu}}$
897 Eq. (45)	$\sqrt{\pi/2}$	$\sqrt{\pi/8}$
979 Table II, heading	$ E_\gamma > 50 \text{ Mev}   E_\gamma > 50 \text{ Mev}$	$ E_\gamma < 50 \text{ Mev}   E_\gamma > 50 \text{ Mev}$
1023 Figure caption		a) $\omega < \omega_H$ , b) $\omega > \omega_H$
1123 Eq. (2)	$\Gamma = \mu_2/\mu_1$	$\Gamma = \mu_2/\mu_1, \mu_\perp = (\mu_1^2 - \mu_2^2)/\mu_1$

**ERRATA TO VOLUME 8**

Page	Reads	Should Read
375 Figure caption	a) positrons of energy up to 0.4 $\epsilon$ , b) positrons of energy up to 0.3 $\epsilon$ .	a) positrons of energy up to 0.3 $\epsilon$ , b) positrons of energy up to 0.4 $\epsilon$ .
816 Beginning of Eq. (8)	$I_2^5 = (4\pi)^2 \dots$	$I_2^2 = (4\pi)^5 \dots$