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KINETIC EQUATIONS FOR A SYSTEM OF PARTICLES INTERACTING WITH LATTICE VIBRATIONS

Iu. L. KLIMONTOVICH and S. V. TEMKO

Moscow State University

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A derivation is given of the kinetic equations for the distribution functions of electrons and vibrations of a crystalline lattice. In the spatially homogeneous case, the equation for the distribution function of the electrons is identical to the kinetic equation in Bloch's theory of electric conductivity. An equation is obtained for the lattice vibrations, and expressions have been found for the frequency and the damping decrement of these vibrations.

IN the present paper, the method of Bogoliubov[1,2] has been applied to the derivation of the kinetic equations for the electron and the oscillations of a crystalline lattice. Several very interesting researches, devoted to the theory of superconductivity, have recently been reported. On the one hand there are the researches of Bardeen, Cooper, and Schrieffer,[3] and on the other, the new method in the theory of superconductivity developed by Bogoliubov.[4] In this connection, there is interest in the study of the kinetic equations for systems of electrons interacting with the lattice vibrations. However, for the purpose of finding a simpler approach to the description of nonequilibrium processes in solids, we limit ourselves in the present note only to the approximation leading to the Bloch theory, and do not take up the question of superconductivity. We note that, in recently published works,[5,6] a derivation has been given of the quantum kinetic equation for electrons interacting with lattice vibrations, in the spatially homogeneous case. This equation is identical to the well-known equation of Bloch theory. The approximate kinetic equations have been obtained by us both for electrons and also for the oscillators of the crystalline lattice in the homogeneous case. Initially, the equations were obtained for the distribution functions of the electrons and the oscillators of the lattice in the classical approximation; these have the form of the Fokker-Planck equations in phase space. Then a derivation was given of the kinetic equations for the corresponding inhomogeneous quantum distribution functions.

Following Fröhlich, we shall write the Hamiltonian function for the system of electrons interacting with the lattice vibrations of a crystal in the form

$$H_{\mathrm{F}} = \sum_{(1\leqslant i\leqslant N)} \frac{\mathbf{p}_i^2}{2m} + \frac{c}{\sqrt{N}} \sum_{(1\leqslant i\leqslant N)} \int \delta(\mathbf{q}-\mathbf{q}_i)\,\mathrm{div}_{\mathbf{q}} \sum_{\mathbf{k}} (\mathbf{Q}_{\mathbf{k}}^{(1)} \times \sin\mathbf{k}\cdot\mathbf{q} + \mathbf{Q}_{\mathbf{k}}^{(2)} \cos\mathbf{k}\cdot\mathbf{q})\,d\mathbf{q} + \sum_{\mathbf{k}} \sum_{j=1}^{2} (\mathbf{P}_{\mathbf{k}}^{(j)2}/2M + M\omega_{\mathbf{k}}^2 \mathbf{Q}_{\mathbf{k}}^{(j)2}/2). \tag{1}$$

Here N is the number of electrons, c is the interaction constant, which has the dimensions of energy; M is the mass of the ions; $\mathbf{Q}_{\mathbf{k}}^{(j)}$, $\mathbf{P}_{\mathbf{k}}^{(j)}$ are the coordinates and the momenta of the lattice oscillators; j = 1, 2; $\omega_k = ks$ is the vibration frequency with wave number k; s is the sound velocity without consideration of the interaction between the electrons and the lattice vibrations of the crystal. Making use of the Hamiltonian (1), we obtained the equations of motion for the electrons and for the oscillators of the lattice. Desiring to set up the kinetic equations for the distribution functions of the electrons and the waves of the crystalline lattice, we introduce the distribution functions of the coordinates and momenta of all the electrons and oscillators of the lattice of the crystal. This distribution function can be normalized since the event, that the coordinates and momenta of the electrons and lattice oscillators have any values at all, is certain. By virtue of the Hamilton equations for the distribution function f thus introduced, the equation

$$\frac{\partial f}{\partial t} + \sum_i \frac{\mathbf{p}_i}{m}\frac{\partial f}{\partial \mathbf{q}_i} + \sum_{\mathbf{k},j}\left(\frac{\mathbf{P}_{\mathbf{k}}^{(j)}}{M}\frac{\partial f}{\partial \mathbf{Q}_{\mathbf{k}}^{(j)}} - M\omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^{(j)}\frac{\partial f}{\partial \mathbf{P}_{\mathbf{k}}^{(j)}}\right) \times \frac{c}{\sqrt{N}} \sum_{\mathbf{k},j,i} (\mathbf{Q}_{\mathbf{k}}^{(j)}\cdot\mathbf{k}) \begin{matrix}\sin\mathbf{k}\,\mathbf{q}_i \\ \cos\mathbf{k}\cdot\mathbf{q}_i\end{matrix}\,\mathbf{k}\frac{\partial f}{\partial \mathbf{p}_i} - \frac{c}{\sqrt{N}} \sum_{\mathbf{k},j,i} \begin{matrix}\cos\mathbf{k}\cdot\mathbf{q}_i \\ -\sin\mathbf{k}\cdot\mathbf{q}_i\end{matrix}\,\mathbf{k}\frac{\partial f}{\partial \mathbf{P}_{\mathbf{k}}^{(j)}} = 0 \tag{2}$$

is obtained. Here and below, the upper function refers to the term with $j = 1$, while the lower refers to $j = 2$. From this equation we can, by integration, obtain a chain of equations for the corresponding distribution functions. We have limited ourselves to the consideration of the equation only for the first distribution functions of the electrons and lattice oscillators $f_1(q, p; t)\, F_1(Q_k^{(j)}, P_k^{(j)}; t)$, and only for one mixed second distribution function $\Phi_2(q \cdot p;\ Q_k^{(j)}, P_k^{(j)};\ t)$. For brevity, we shall not write out the expansion here. (Where possible, we shall omit the arguments of the distribution functions in what follows.) To obtain a closed set of approximate equations according to the work of Bogoliubov and Gurov,[2] we have made use of the following approximations of the second distribution functions:

$$\Phi_2(\mathbf{q}, \mathbf{p}; \mathbf{Q}_\mathbf{k}, \mathbf{P}_\mathbf{k}; t) = f_1(\mathbf{q}, \mathbf{p}; t)\, F_1(\mathbf{Q}_\mathbf{k}, \mathbf{P}_\mathbf{k}; t) + g(\mathbf{q}, \mathbf{p}; \mathbf{Q}_\mathbf{k}, \mathbf{P}_\mathbf{k}; t),$$

where g is the correlation function that governs the interaction of the electrons with the lattice vibrations. The third distribution functions are represented in the form of products of the corresponding first distribution functions. Here the interaction between the electrons and the lattice vibrations is assumed to be weak, and we can therefore regard the function of the correlation deviation g to be small in comparison with the products of the first distribution functions. In the approximation under consideration, the equations for f_1, F_1 and g take on the following forms:

$$\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial f_1}{\partial \mathbf{q}} + \frac{c}{\sqrt{N}}\sum_{j,\mathbf{k}}\int (\mathbf{Q}_\mathbf{k}^{(j)}\cdot\mathbf{k}) \begin{matrix}\sin \mathbf{k}\cdot\mathbf{q}\\ \cos \mathbf{k}\cdot\mathbf{q}\end{matrix}\, F_1(\mathbf{Q}_\mathbf{k}, \mathbf{P}_\mathbf{k}; t) \times d\mathbf{Q}_\mathbf{k} d\mathbf{P}_\mathbf{k}\, \mathbf{k}\frac{\partial f_1}{\partial \mathbf{p}} = -\frac{c}{\sqrt{N}}\sum_{j,\mathbf{k}}\int (\mathbf{k}\cdot\mathbf{Q}_\mathbf{k}^{(j)}) \begin{matrix}\sin \mathbf{k}\cdot\mathbf{q}\\ \cos \mathbf{k}\cdot\mathbf{q}\end{matrix}\, \mathbf{k}\frac{\partial g}{\partial \mathbf{p}}\, d\mathbf{Q}_\mathbf{k} d\mathbf{P}_\mathbf{k}; \tag{3}$$

$$\frac{\partial F_1}{\partial t} + \sum_j \left(\frac{\mathbf{P}_\mathbf{k}^{(j)}}{M}\frac{\partial F_1}{\partial \mathbf{Q}_\mathbf{k}^{(j)}} - M\omega_\mathbf{k}^2 \mathbf{Q}_\mathbf{k}^{(j)}\frac{\partial F_1}{\partial \mathbf{P}_\mathbf{k}^{(j)}}\right) - \frac{c}{\sqrt{N}}\frac{1}{v}\sum_j \int \begin{matrix}\cos \mathbf{k}\cdot\mathbf{q}\\ -\sin \mathbf{k}\cdot\mathbf{q}\end{matrix}\, f_1(\mathbf{q}; \mathbf{p}; t)\, d\mathbf{q}\, d\mathbf{p}\, \mathbf{k}\frac{\partial F_1}{\partial \mathbf{P}_\mathbf{k}^{(j)}} = \frac{c}{\sqrt{N}}\frac{1}{v}\sum_j \int \begin{matrix}\cos \mathbf{k}\cdot\mathbf{q}\\ -\sin \mathbf{k}\cdot\mathbf{q}\end{matrix}\, \mathbf{k}\frac{\partial g}{\partial \mathbf{P}_\mathbf{k}^{(j)}}\, d\mathbf{q}\, d\mathbf{p}; \quad v = V/N; \tag{4}$$

$$\frac{\partial g}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial g}{\partial \mathbf{q}} + \sum_j \left(\frac{\mathbf{P}_\mathbf{k}^{(j)}}{M}\frac{\partial g}{\partial \mathbf{Q}_\mathbf{k}^{(j)}} - M\omega_k^2 \mathbf{Q}_\mathbf{k}^{(j)}\frac{\partial g}{\partial \mathbf{P}_\mathbf{k}^{(j)}}\right) = \frac{c}{\sqrt{N}}\sum_j \begin{matrix}\cos \mathbf{k}\cdot\mathbf{q}\\ -\sin \mathbf{k}\cdot\mathbf{q}\end{matrix}\, \mathbf{k}\frac{\partial F_1}{\partial \mathbf{P}_\mathbf{k}^{(j)}} f_1 - \frac{c}{\sqrt{N}}\sum_j (\mathbf{k}Q_\mathbf{k}^{(j)}) \begin{matrix}\sin \mathbf{k}\cdot\mathbf{q}\\ \cos \mathbf{k}\cdot\mathbf{q}\end{matrix}\, \mathbf{k}\frac{\partial f_1}{\partial \mathbf{p}} F_1. \tag{5}$$

If, with the aid of Eq. (5), we express the function g in terms of the first distribution function f_1, F_1 and substitute the expressions thus obtained in Eqs. (3) and (4), then we can get a set of equations for f_1 and F_1, which serve for the description of random processes in the crystalline lattice with account of the previous history. General consideration of such a type of processes was carried out by Khinchin.[7] However, the solutions of these equations will naturally depend not only on the initial values of the functions f_1, F_1, but also on the initial values of the correlation function g. Inasmuch as the initial value of g is usually unknown, we are obliged to consider the system of approximate kinetic equations for the functions f_1 and F_1, the solutions of which must depend only on the initial values of the functions f_1 and F_1. Such a system of approximate equations is valid only over time intervals that exceed the correlation time. This corresponds to the fact that, in the solution of the equation for g, the time dependence of the correlation function is determined by specifying the first functions f_1 and F_1 at this same instant of time.

Wishing to obtain from such a system of equations for f_1 and F_1 a separate approximate equation for the electron distribution function f_1 with accuracy up to c^2, we assume that at the initial instant the lattice vibrations are in equilibrium, while the states of the electrons depart little from the equilibrium state. In this case we must, by way of the initial distribution of the oscillators of the lattice, substitute in Eq. (5) the solution of the equation for F_1 at $c = 0$; this has the following form:

$$F_1^{(0)} = B \exp\left\{-\sum_j (\mathbf{P}_\mathbf{k}^{(j)2}/M + M\omega_k^2 \mathbf{Q}_\mathbf{k}^{(j)2})/2\varkappa T\right\}.$$

In transforming the left side of the equation for the function f_1, (by virtue of the required accuracy) we require a more detailed solution of the equation for F_1, namely

$$F_1 = F_1^{(0)}(\mathbf{Q}_\mathbf{k}^{(j)0}, \mathbf{P}_\mathbf{k}^{(j)0}; t_0) + \frac{c}{\sqrt{N}}\frac{1}{v} \times \sum_{l=1}^{2}\int_0^{t-t_0}\int \begin{matrix}\cos \mathbf{k}\cdot\mathbf{q}\\ -\sin \mathbf{k}\cdot\mathbf{q}\end{matrix}\, f_1(\mathbf{q}, \mathbf{p}; t-\theta)\, d\mathbf{q}\, d\mathbf{p} \left(\mathbf{k}\frac{\partial F_1^{(0)}}{\partial \mathbf{P}_\mathbf{k}^{(l)}}\right)_0 d\theta. \tag{6}$$

Here

$$\mathbf{Q}_\mathbf{k}^{(j)0} = \mathbf{Q}_\mathbf{k}^{(j)} \cos \omega_\mathbf{k}\theta - \frac{\mathbf{P}_\mathbf{k}^{(j)}}{M\omega_k}\sin \omega_k\theta,$$

$$\mathbf{P}_\mathbf{k}^{(j)0} = M\omega_k \mathbf{Q}_\mathbf{k}^{(j)} \sin \omega_k\theta + \mathbf{P}_\mathbf{k}^{(j)} \cos \omega_k\theta,$$

$(\ldots)_0$ means that $Q_k^{(j)0}, P_k^{(j)0}$ must be substituted for $Q_k^{(j)}, P_k^{(j)}$ after differentiation. After the elimination of the functions F_1 and g from Eq. (3), we obtain the kinetic equation for the distribution function of the electrons, interacting with the lattice vibrations of the crystal, in the form of the Fokker-Planck equation in phase space:[8]

$$\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}}{m}\cdot\frac{\partial f_1}{\partial \mathbf{q}} - \frac{c^2}{4\pi M s^2}\Delta_{\mathbf{q}}$$

$$\times \int\limits_{|\mathbf{q}-\mathbf{q}'|\leqslant s(t-t_0)} \frac{\frac{\partial}{\partial \mathbf{q}'} f_1(\mathbf{q}', \mathbf{p}'; t - |\mathbf{q}-\mathbf{q}'|/s)}{|\mathbf{q}-\mathbf{q}'|} d\mathbf{q}' d\mathbf{p}' \frac{\partial f_1}{\partial \mathbf{p}}$$

$$= \frac{\partial}{\partial \mathbf{p}} B(s; \mathbf{p}) \frac{\partial}{\partial \mathbf{p}} f_1(\mathbf{q}, \mathbf{p}; t) + \frac{\partial}{\partial \mathbf{p}} \mathbf{A}(s; \mathbf{p}) f_1(\mathbf{q}, \mathbf{p}; t). \quad (7)$$

Here A and B are the Fokker-Planck coefficients for the given system:

$$B(s, \mathbf{p}) = \frac{c^2 \varkappa T v}{8\pi^2 M s^2} \int \delta\left(\omega_k - \frac{\mathbf{k}\cdot\mathbf{p}}{m}\right) \mathbf{k}^2 d\mathbf{k},$$

$$\mathbf{A}(s; \mathbf{p}) = \frac{c^2 v}{8\pi^2 M s^2} \int \omega_k \mathbf{k} \delta\left(\omega_k - \frac{\mathbf{k}\cdot\mathbf{p}}{m}\right) d\mathbf{k}.$$

We note that the coefficients of "diffusion" and "friction" in Eq. (7) are different from zero only upon satisfaction of the radiation condition. It is natural that, in the classical approximation, the Fokker-Planck coefficients A and B diverge for large wave numbers k. However, we can break off the integration over k at some value k_{max} which can be chosen from the Debye temperature Θ: $k_{max} = \kappa\Theta/\hbar s$; κ is the Boltzmann constant. After integration over k, the coefficients A and B take the following form:

$$A^\alpha = \frac{c^2 m v}{4\pi M s^2 p}\left(\frac{\varkappa\theta}{\hbar s}\right) D(x); \quad B^{\alpha\beta} = \frac{c^2 m \varkappa T v}{4\pi M s^2 p}\left(\frac{\varkappa\theta}{\hbar s}\right) D(x),$$

where

$$D(x) = \frac{1}{x^4}\int_0^x \frac{y^4 dy}{e^y - 1}, \quad x \doteq \frac{\Theta}{T}.$$

In the equilibrium case, Eq. (7) is satisfied by the Maxwell distribution. The region of integration on the left side of Eq. (7) is bounded by the condition $|\mathbf{q}-\mathbf{q}'| \leq s(t-t_0)$. This limitation of the integration over $\mathbf{q}'$ arises from the fact that the problem is considered when, at the initial moment $t = t_0$, the distribution functions of the electrons and the oscillators of the lattice are given. Therefore, the only electrons that can have an effect on the change of the function f_1 at the point $\mathbf{q}$ and time $t > t_0$ are those removed from $\mathbf{q}$ by the distance $|\mathbf{q}-\mathbf{q}'| \leq s(t-t_0)$, since in this case the electrons interact only through vibrations of the crystalline lattice.

The approximate kinetic equation for the distribution function of the oscillators of the lattice is similarly constructed. Only now we must consider the initial distribution of the electrons at equilibrium. On the other hand, the initial state of the lattice vibrations here must be assumed to differ slightly from the equilibrium state. As a result, the equation for the distribution function F_1 has the form:

$$\frac{\partial F_1}{\partial t} + \sum_j \frac{\mathbf{P}_{\mathbf{k}}^{(j)}}{M} \frac{\partial F_1}{\partial \mathbf{Q}_{\mathbf{k}}^{(j)}} - \sum_j \left\{ M\Omega_k^2 \mathbf{Q}_{\mathbf{k}}^{(j)} \right.$$

$$\left. + \frac{c^2}{M\omega_k} \operatorname{Re} I_2 \left(\mathbf{k} \frac{\partial}{\partial \mathbf{Q}_{\mathbf{k}}^{(j)}}\right) \mathbf{k} \right\} \frac{\partial F_1}{\partial \mathbf{P}_{\mathbf{k}}^{(j)}}$$

$$+ \sum_j \int \int_0^{t-t_0} U^{(j)}(\mathbf{Q}_{\mathbf{k}}'; t-\theta)\, d\theta\, d\mathbf{Q}_{\mathbf{k}}' \cdot \mathbf{k} \frac{\partial F_1}{\partial \mathbf{P}_{\mathbf{k}}^{(j)}}$$

$$= c^2 \sum_j \left\{ \operatorname{Im} I_2 \mathbf{k} \frac{\partial}{\partial \mathbf{P}_{\mathbf{k}}^{(j)}} \mathbf{k} \frac{\partial F_1}{\partial \mathbf{P}_{\mathbf{k}}^{(j)}} + \operatorname{Im} I_1 \mathbf{k} \frac{\partial}{\partial \mathbf{P}_{\mathbf{k}}^{(j)}} \left(\frac{\mathbf{k}\cdot\mathbf{P}_{\mathbf{k}}^{(j)}}{M\omega_k}\right) F_1 \right\}. \quad (8)$$

Here

$$\Omega_k^2 = \omega_k^2 + (c^2 k^2/M) \operatorname{Re} I_1;$$

$$I_1 = \int \frac{\mathbf{k} \frac{\partial f_1^{(0)}}{\partial \mathbf{p}} d\mathbf{p}}{(\mathbf{k}\cdot\mathbf{p}/m) - \omega_k}; \quad I_2 = \int \frac{f_1^{(0)} d\mathbf{p}}{(\mathbf{k}\cdot\mathbf{p}/m) - \omega_k};$$

$$U^{(j)} = -c^2 \int (\mathbf{k}\cdot\mathbf{Q}_{\mathbf{k}}^{(j)\prime}) F_1(\mathbf{Q}_{\mathbf{k}}', \mathbf{P}_{\mathbf{k}}', t-\theta) \mathbf{k} \frac{\partial f_1^{(0)}}{\partial \mathbf{p}}$$

$$\times \sin\frac{\mathbf{k}\cdot\mathbf{p}}{m}\theta \cdot d\mathbf{p}\, d\mathbf{P}_{\mathbf{k}}';$$

$f_1^{(0)}$ is the equilibrium distribution function of the conduction electrons. In the case of thermal equilibrium, $F_1^{(0)}$ does not make the right side of Eq. (8) vanish. Because of the presence of the interaction, the stationary distribution of the waves of the lattice, by virtue of Eq. (6), takes the form $F_1(Q_k, P_k) = F_1^{(0)}(Q_k, P_k) + c^2 F_1^{(1)}(Q_k, P_k)$, which agrees with the assumption of Peierls.[9]

Upon the derivation of the quantum kinetic equation for electrons and phonons, we start out from the equation for the quantum distribution function (the density matrix in the mixed representation) in the form set forth by Wigner. As a Hamiltonian, we again use Eq. (1). In the self-consistent approximation for the quantum distribution function of the conduction electrons f_1, we find:

$$\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial f_1}{\partial \mathbf{q}} + \frac{ic^2}{4\pi s^2 M\hbar}\frac{1}{(2\pi)^3}\Delta_{\mathbf{q}}$$

$$\times \int \left[\frac{f_1\left(\mathbf{q}', \mathbf{p}'; t - \left|\mathbf{q}-\mathbf{q}'-\frac{1}{2}\hbar\tau\right| / s\right)}{\left|\mathbf{q}-\mathbf{q}'-\frac{1}{2}\hbar\tau\right|} \right.$$

$$\left. - \frac{f_1(\mathbf{q}', \mathbf{p}'; t - |\mathbf{q}-\mathbf{q}'+\hbar\tau/2|/s)}{|\mathbf{q}-\mathbf{q}'+\hbar\tau/2|} \right]$$

$$\times e^{i\tau(\eta-\mathbf{p})} f_1(\mathbf{q}, \eta; t)\, d\tau\, d\eta\, d\mathbf{p}' d\mathbf{q}' = 0. \quad (9)$$

The region of integration over $\mathbf{q}'$ in this equation is limited by the condition $|\mathbf{q}-\mathbf{q}' \pm \hbar\tau/2| \leq s(t-t_0)$, where the upper sign refers to the first, and the lower, to the second, integral. For $F_1^{(0)}$ we have used the expression for the quantum distribution function of oscillators obtained in reference 10. Account of correlation permits us to compute the change in the distribution function f_1 at the expense of the "collision" of electrons with the lattice waves, which can be represented in the form of a set of three expressions. The first of these is similar to the diffusion term in Eq. (7) and can be written in the form

$$\frac{\pi c^2}{s^2\hbar MN}\sum_{\mathbf{k}}\varkappa T_{\mathbf{k}}\Big\{\delta\Big(\frac{p^2}{2m}-\frac{(\mathbf{p}+\hbar\mathbf{k})^2}{2m}-\hbar\omega_{\mathbf{k}}\Big)\times\Big[f_1(\mathbf{p}+\hbar\mathbf{k},\mathbf{q};t)-f_1(\mathbf{p},\mathbf{q};t)\Big]$$
$$+\delta\Big(\frac{p^2}{2m}-\frac{(\mathbf{p}+\hbar\mathbf{k})^2}{2m}+\hbar\omega_{\mathbf{k}}\Big)[f_1(\mathbf{p}+\hbar\mathbf{k},\mathbf{q};t)-f_1(\mathbf{p},\mathbf{q};t)]\Big\},$$

where

$$\varkappa T_{\mathbf{k}}=\frac{1}{2}\hbar\omega_{\mathbf{k}}+\frac{\hbar\omega_k}{\exp\{\hbar\omega_k/\varkappa T\}-1};$$

the second is similar to the term of systematic friction and has the form

$$\frac{\pi c^2}{s^2\hbar MN}\sum_{\mathbf{k}}\frac{\hbar\omega_k}{2}\Big\{\delta\Big(\frac{p^2}{2m}-\frac{(\mathbf{p}+\hbar\mathbf{k})^2}{2m}+\hbar\omega_k\Big)\times\Big[f_1(\mathbf{p}+\hbar\mathbf{k},\mathbf{q};t)+f_1(\mathbf{p},\mathbf{q};t)\Big]$$
$$-\delta\Big(\frac{p^2}{2m}-\frac{(\mathbf{p}+\hbar\mathbf{k})^2}{2m}-\hbar\omega_k\Big)[f_1(\mathbf{p}+\hbar\mathbf{k},\mathbf{q};t)+f_1(\mathbf{p},\mathbf{q};t)]\Big\};$$

while the third is due to exchange effects and has no classical analogue:

$$\frac{c^2}{vs^2\hbar MN(2\pi)^3}\sum_{\mathbf{k}}\iint_0^\infty\omega_k\sin\omega_k\theta\sin\mathbf{k}\Big(\mathbf{q}-\mathbf{q}'-\frac{\boldsymbol{\eta}}{m}\theta\Big)\times\Big[\delta\Big(\boldsymbol{\eta}-\mathbf{p}+\frac{\hbar\mathbf{k}}{2}\Big)-\delta\Big(\boldsymbol{\eta}-\mathbf{p}-\frac{\hbar\mathbf{k}}{2}\Big)\Big]f_1$$
$$\times\Big(\frac{\mathbf{q}+\mathbf{q}'-\frac{\boldsymbol{\eta}}{m}\theta}{2}+\frac{\hbar\boldsymbol{\tau}}{4},\ \boldsymbol{\pi}';t\Big)f_1\Big(\frac{\mathbf{q}+\mathbf{q}'-\frac{\boldsymbol{\eta}}{m}\theta}{2}-\frac{\hbar\boldsymbol{\tau}}{4},\boldsymbol{\pi}'';t\Big)$$
$$\times\exp\Big\{i\Big(\mathbf{q}-\mathbf{q}'-\frac{\boldsymbol{\eta}}{m}\theta\Big)\frac{\pi'-\pi''}{\hbar}+i\tau\Big(\frac{\pi'+\pi''}{2}-\boldsymbol{\eta}\Big)\Big\}d\theta\,d\mathbf{q}'\,d\boldsymbol{\pi}'\,d\boldsymbol{\pi}''\,d\boldsymbol{\tau}\,d\boldsymbol{\eta}.$$

Joining these together, after substitution in the right side of Eq. (9), we find the quantum kinetic equation for the electrons in the inhomogeneous case. For a homogeneous distribution, the exchange term is materially simplified, and the equation for the distribution function of the electrons takes the following form:

$$\frac{\partial w(p;t)}{\partial t}=\frac{\pi c^2}{s^2\hbar MN}\sum_{\mathbf{k}}\Big\{\delta\Big(\frac{p^2}{2m}-\frac{(\mathbf{p}+\hbar\mathbf{k})^2}{2m}-\hbar\omega_k\Big)\Big[\overline{N}_{\hbar\omega_k}w$$
$$\times(\mathbf{p}+\hbar\mathbf{k})(1-w(\mathbf{p}))-(\overline{N}_{\hbar\omega_k}+\hbar\omega_k)w(\mathbf{p})(1-w(\mathbf{p}+\hbar\mathbf{k}))\Big]$$
$$+\delta\Big(\frac{p^2}{2m}-\frac{(\mathbf{p}+\hbar\mathbf{k})^2}{2m}+\hbar\omega_k\Big)[(\overline{N}_{\hbar\omega_k}+\hbar\omega_k)w(\mathbf{p}+\hbar\mathbf{k})$$
$$\times(1-w(\mathbf{p}))-\overline{N}_{\hbar\omega k}w(\mathbf{p})(1-w(\mathbf{p}+\hbar\mathbf{k}))]\Big\}.\qquad(10)$$

Here

$$\overline{N}_{\hbar\omega k}=\hbar\omega_k(e^{\hbar\omega k/\varkappa T}-1)^{-1};\ w(\mathbf{p};t)=\frac{(2\pi\hbar)^3}{v}f_1(\mathbf{p};t).$$

This equation is identical with the well known equation of Bloch theory.[9,11] Relative to the quantum kinetic equation for the distribution function of the phonons F_1, we note that it preserves the form of Eq. (8), with this one difference, that the expressions for the potential (8) and for the integrals I_1 and I_2 are changed. From Eq. (8) we get the averaged equation of the vibrations of the crystalline lattice, which, for a homogeneous distribution of electrons, has the following form:

$$\ddot{\overline{\mathbf{Q}}}_{\mathbf{k}}^{(j)}+2\gamma_k\dot{\overline{\mathbf{Q}}}_{\mathbf{k}}^{(j)}+\Omega_k^2\overline{\mathbf{Q}}_{\mathbf{k}}^{(j)}=0,\qquad(11)$$

where

$$\overline{\mathbf{Q}}_k^{(j)}=\int\mathbf{Q}_{\mathbf{k}}^{(j)}F_1d\mathbf{Q}_{\mathbf{k}}^{(j)}d\mathbf{P}_{\mathbf{k}}^{(j)};\ 2\gamma_k=\frac{c^2k^2}{M\omega_k}\operatorname{Im}I_1.$$

In the case of complete Fermi degeneracy for the electrons, the frequency of the characteristic vibrations Ω_{k} and the damping decrement γ_{k} of the lattice waves are easily computed:

$$\Omega_k^2=\omega_k^2-\frac{3c^2m}{p_0^2M}k^2;\ \gamma_k=\frac{3}{8}\pi\Big(\frac{c}{\varepsilon_0}\Big)^2\frac{p_0}{M}k,\qquad(12)$$

where ϵ_0, p_0 are the limiting values of the energy and momentum for the Fermi distribution.

The value for Ω_{k} corresponds to the expression obtained in the work of Fröhlich.[12] For the estimate of the value of the damping γ_{k}, we can use, for example, the tabulated data for (c/ϵ_0) given by Bethe and Sommerfeld.[9]

We note that in a number of cases it is not possible to obtain separate equations for f_1, F_1 from the approximate (in the sense given above) system of equations for the distribution functions of the electrons and waves of the crystalline lattice. We must then solve simultaneously the set of equations for these functions. In our treatment, we started from the Hamiltonian of Fröhlich, in which neither the periodic field of the ions nor the interaction between the phonons and the Coulombic interaction between the conduction electrons were considered. Employing the results of the present note, we hope in the future to consider more accurate kinetic equations for the electrons and the vibrations of the crystalline lattice.

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