INTERACTION BETWEEN A CHARGED CURRENT-CONDUCTING JET MOVING IN A CIRCLE AND A FERRITE

A. N. DIDENKO

Moscow State University

Submitted to JETP editor January 18, 1958; resubmitted May 22, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) 35, 655-661 (September, 1958)

The radiation spectrum of a current-conducting jet which moves in a circle in a homogeneous space is considered. Asymptotic expressions for Bessel and Neumann functions, which yield good approximations even at relatively low values of the indices and arguments, are used to study the radiation spectrum and the transverse forces exerted by the ferrite in cases of non-uniform media.

1. INTRODUCTION. METHOD OF CALCULATION

THE interaction between current-conducting jets, moving with high velocities, and a ferrite was first considered by Morozov.[1] The ramifications of this problem were noted by this author: in studying earlier cases of practical importance the interaction could be analyzed in a quasi-stationary approximation. It now becomes apparent, however, that there are a number of physical problems in which one deals with objects which are not true currents or charges, and which move as a unit with velocities greater than the phase velocity of light in many materials (high-current accelerators, new methods of generating radio waves). In reference 1 the analysis was made assuming rectilinear motion of the jet. In this case the "trajectories" are planes parallel to one of the Cartesian coordinate planes. The entire space was assumed to be made up of layers characterized by different values of ϵ and μ, all parallel to the same coordinate plane.

However, great interest attaches to the case in which the jet moves in a circle. In this case the "trajectories" are cylindrical surfaces and the space is divided into cylindrical layers characterized by different values of ϵ and μ, which are coaxial with the trajectory cylinders (this situation is considered in greater detail in Sec. 3). An investigation of this configuration is of interest in its own right from the point of view of general radiation theory because at certain critical velocities the total radiation loss is determined by the Cerenkov effect as well as the finite radius of curvature of the trajectory.

In the present paper we consider several cases in which a charged current-conducting jet moving in a circle interacts with a ferrite medium or media. We assume that the different parts of space are media characterized by different ϵ and μ. We assume that a charged current-conducting jet moves in the aximuthal direction in a circle of radius a with a velocity $\mathbf{v}(0, v, V_z)$ in one of these media; it is also assumed that the jet is infinitesimally thin and unbounded in the z direction and that it traces out a cylinder of radius a in its motion (cf. Fig. 1). We denote the strength of the current which flows in the jet by I_0; the linear charge density is denoted by ρ. The current components are given by the following expressions:

$$j_r = 0;\ j_\varphi = \rho v \frac{\delta(r-a)}{r}\delta(\varphi - \omega_0 t);$$
$$j_z = I\frac{\delta(r-a)}{r}\delta(\varphi-\omega_0 t)\ \left(\omega_0 = \frac{v}{a}\right). \tag{1}$$

It can be shown that the electromagnetic field described by Maxwell equations can be broken up into two independent fields: one of these is determined by the total current $I = I_0 + \rho V_z$ and the other by the charge ρ. As in reference 1, we call the first the I-field and the second the ρ-field.

Substituting Eq. (1) in Maxwell equations and solving the system of homogeneous equations with appropriate boundary conditions (cf., e.g., reference 2) we find the field components and the forces which act on the jet in several particular cases.

In order to obtain definite results we assume that $\epsilon(\omega)$ and $\mu(\omega)$ are real in all media; this assumption is not entirely justified since it is necessary to take account of absorption in considering dispersive media.

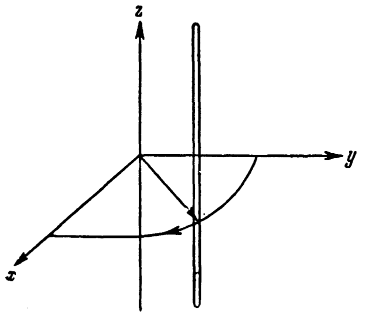

FIG. 1

2. MOTION IN HOMOGENEOUS SPACE

We first consider the motion of a charged current filament (infinite along the z axis) in a medium characterized by ϵ and μ which fills the entire space (Fig. 1). Carrying out the calculation in accordance with the scheme given above, we find the field components and the radiation losses for the I-field and the ρ-field; the latter is taken as the scalar product of the force which acts on the jet and its velocity, with sign reversed:[3]

$$W = -\mathbf{v}\cdot\mathbf{F} = W^I + W^\rho = \sum_{n=1}^{\infty} W_n^I + \sum_{n=1}^{\infty} W_n^\rho ,$$

where

$$W_n^\rho = (2\pi\mu k\rho^2 v^2 / c)\, J_n'^2 (\sqrt{\varepsilon\mu} ka), \tag{2}$$

$$W_n^I = (2\pi\mu k I^2 / c)\, J_n^2(\sqrt{\varepsilon\mu} ka). \tag{3}$$

It is easy to show that $-\mathbf{v}\cdot\mathbf{F} = \partial U/\partial t$ where $\partial U/\partial t$ is the energy flux computed from the Poynting theorem. A discussion of this result, which is not completely obvious for the jet case, is given in reference 1.

We now consider the motion of the jet in empty space ($\epsilon = \mu = 1$). In the nonrelativistic case it is easily shown that the maximum radiation of the I-field and the ρ-field occurs at the first harmonic (dipole radiation). In the relativistic case, which is of greatest interest, W_n^ρ and W_n^I are expressed in terms of Bessel functions $J_n(x)$ and their derivatives for which $n \sim x$ $(n-x)/n = \xi$; $\xi \ll 1$). Thus, to study the radiation spectra we introduce the asymptotic expansion for $J_n(x)$ and $J'(x)$; these can give good approximations even at small values of the indices and arguments ($n \sim x$). Without considering this procedure in detail, we may note that the method given in reference 4 gives a comparatively simple way of finding the following asymptotic expressions:

$$J_n(x) = \frac{1}{\pi} \sqrt{\frac{2\xi}{3}}\, K_{1/3}\left[\frac{n}{3}(2\xi)^{3/2}\right] e^{-\sigma},$$

$$J_n'(x) = \frac{2\xi}{\pi\sqrt{3}}\, K_{2/3}\left[\frac{n}{3}(2\xi)^{3/2}\right] e^{-\sigma}, \tag{4}$$

$$\xi = \frac{n-x}{n} \ll 1; \quad \sigma = \frac{B_1}{2n} - \frac{B_2}{12n^3} + \ldots;$$

where K is the MacDonald function and B_i is the Bernuolli number (cf. reference 5). The asymptotic expressions for the Neumann functions are given in this same paper; these forms will be required for analysis of other particular cases and are better than the asymptotic expressions given by Fock.[6]

Substituting the expressions for $J_n(x)$ and $J'_n(x)$ from Eq. (4) in Eqs. (2) and (3) we obtain the following expressions for the radiation of the I-field and the ρ-field:

$$W^\rho = \sum_{n=1}^{\infty} \frac{2}{3\pi} \frac{\rho^2 n \beta^3 c}{a} \left(\frac{mc^2}{E}\right)^4 K_{2/3}^2 \left[\frac{n}{3}\left(\frac{mc^2}{E}\right)^3\right] e^{-2\sigma}, \tag{5}$$

$$W^I = \sum_{n=1}^{\infty} \frac{2}{3\pi} \frac{I^2 n \beta}{ac} \left(\frac{mc^2}{E}\right)^2 K_{1/3}^2 \left[\frac{n}{3}\left(\frac{mc^2}{E}\right)^3\right] e^{-2\sigma}. \tag{6}$$

It is apparent from Eqs. (5) and (6) that in the relativistic case there is a difference in the spectral energy distribution for the I-field and the ρ-field. From Eq. (5) it follows that the radiation intensity of the n-th harmonic of the ρ-field is inversely proportional to $n^{1/3}$ when $n \ll 3(E/mc^2)^3$ and falls off exponentially when $n \gg 3(E/mc^2)^3$, i.e., the strongest radiation in the relativistic case is at the first harmonic. We may note that in the relativistic case the total radiation intensity of the ρ-field is independent of the energy of the jet. On the other hand, the radiation intensity of the I-field, as is apparent from Eq. (6), is directly proportional to $n^{1/3}$ when $n \ll 3(E/mc^2)^3$, is a maximum when $n = 3(E/mc^2)^3$ and falls off exponentially when $n \gg 3(E/mc^2)^3$. For the high harmonics $n \gg 1$ (in the present case these harmonics make the main contribution to the total radiation energy) and we may assume that the radiation frequency ω is continuous. Thus, the summation over n is replaced by integration; using the value of the integral computed by Klepikov:

$$\int_0^\infty K_\nu(x) K_\rho(x) x^{\mu-1} dx = \frac{2^{\mu-3}}{\Gamma(\mu)} \Gamma\left(\frac{\mu+\nu+\rho}{2}\right) \Gamma\left(\frac{\mu-\nu+\rho}{2}\right) \times \Gamma\left(\frac{\mu+\nu-\rho}{2}\right) \Gamma\left(\frac{\mu-\nu-\rho}{2}\right) \quad (\mathrm{Re}\,\mu > |\mathrm{Re}(\nu \pm \rho)|), \tag{7}$$

we find the total radiation intensity per unit length of a jet characterized by a current I :

$$W^I = (2/\sqrt{3})\,(I^2\beta/ac)\,(E/mc^2)^4. \quad (8)$$

It follows from Eq. (8) that the radiation of the I-field of a jet and the radiation of a relativistic electron which moves in a circle are strong functions of energy. Apparently, this is explained by the different directivity of the radiation of the I-field and the ρ-field. A similar result has been obtained by Morozov[1] for a current filament which oscillates in a direction perpendicular to its length.

3. INHOMOGENEOUS SPACES

We now consider the case in which the space is filled with different media. As has already been noted, the portions of space filled by media with ϵ and μ form coaxial cylindrical surfaces of different radii (cf. Fig. 2). In one of the media a charged current-conducting jet, infinite along the z axis, describes a circle of radius a and traces out a coaxial cylinder of radius a in its motion. As before we denote the total current of the jet by I and the linear density by ρ.

We consider three cases below:

(1) a jet which moves in a circle of radius a inside a cylinder of radius a_1 ($a < a_1$) characterized by ϵ_1 and μ_1, surrounded by a medium described by ϵ_2 and μ_2 (Fig. 2, I);

(2) a jet which moves in a circle of radius a in a medium described by ϵ_2 and μ_2 around a ferrite cylinder of radius a_1 characterized by ϵ_1 and μ_1 ($a_1 < a$) (Fig. 2, II);

(3) a jet which moves in a circle of radius a inside a curved channel in a medium characterized by ϵ_2 and μ_2, formed by cylindrical surfaces of radii a_1 and a_2 ($a_1 < a < a_2$); the channel is surrounded by media described by $\epsilon_{1,3}$ and $\mu_{1,3}$ (Fig. 2, III).

If we neglect absorption the radiation intensity of the ρ-field is given by the following:

$$W_i^\rho = \sum_{n=1}^{\infty} W_n^\rho\, \varphi_{n,i}^\rho \quad (i = 1, 2, 3),$$

$$\varphi_{n,1}^\rho = \frac{\chi}{\alpha_1^2 + \beta_1^2}\,\frac{4}{\pi^2 k_1 k_2 a_1^2};$$

$$\varphi_{n,2}^\rho = \frac{1}{J_n'^2(k_s a)}\,\frac{[\alpha_1 N_n'(k_s a) - \beta_1 J_n'(k_s a)]^2}{\alpha_1^2 + \beta_1^2};$$

$$\varphi_{n,3}^\rho = \frac{1}{J_n'^2(k_s a)}\,\frac{4\chi}{\pi^2 k_1 k_2 a_2^2}\,\frac{[\alpha_1 N_n'(k_s a) - \beta_1 J_n'(k_s a)]^2}{A^2 + B^2};$$

$$\alpha_1 = J_n(k_1 a_1)\,J_n'(k_2 a_1) - \chi J_n(k_2 a_1)\,J_n'(k_1 a_1);$$

$$\beta_1 = J_n(k_1 a_1)\,N_n'(k_2 a_1) - \chi N_n(k_2 a_1)\,J_n'(k_1 a_1);$$

$$\gamma_1 = N_n(k_1 a_1)\,J_n'(k_2 a_1) - \chi J_n(k_2 a_1)\,N_n'(k_1 a_1); \quad (9)$$

$$\delta_1 = N_n(k_1 a_1)\,N_n'(k_2 a_1) - \chi N_n(k_2 a_1)\,N_n'(k_1 a_1);$$

$$\chi = \sqrt{\varepsilon_2\mu_1/\varepsilon_1\mu_2}; \quad k_{1,2} = \sqrt{\varepsilon_{1,2}\mu_{1,2}}\,k; \quad k_s = \sqrt{\varepsilon_s \mu_s}\,k;$$

$$A = \alpha_1\beta_2 - \beta_1\alpha_2; \quad B = \alpha_1\delta_2 - \beta_1\gamma_2;$$

α_2, β_2, γ_2 and δ_2 are obtained by substituting $k_1 \to k_3$; $a_1 \to a_2$ in the expressions for α_1, β_1, γ_1 and S_1; the subscript "s" on ϵ, μ, and k indicate values for the media in which the jet moves; the subscript "i" refers to the particular case considered in this section.

In a similar way we obtain expressions for the radial forces exerted on the jet by the ρ-fields (below we call these transverse forces):

$$F_{r,i}^\rho = \sum_{n=1}^{\infty} F_{r,n}^\rho f_{n,i}^\rho \quad (i = 1, 2, 3), \quad (10)$$

where

$$F_{r,n}^\rho = \frac{2\pi\mu_s k_s \rho^2 v^2}{c^2}\left(1 - \frac{1}{\varepsilon_s\mu_s\beta^2}\right) J_n(k_s a)\,N_n'(k_s a);$$

$$f_{n,1}^\rho = -\frac{J_n'(k_s a)}{N_n'(k_s a)}\,\frac{\alpha_1\gamma_1 + \beta_1\delta_1}{\alpha_1^2 + \beta_1^2};$$

$$f_{n,2}^\rho = -\left\{\frac{[\alpha_1 N_n(k_s a) - \beta_1 J_n(k_s a)]\,[\alpha_1 J_n'(k_s a) + \beta_1 N_n'(k_s a)]}{J_n(k_s a)\,N_n'(k_s a)\,(\alpha_1^2 + \beta_1^2)} + 1\right\};$$

$$f_{n,3}^\rho = -\left\{\frac{[\alpha_1 N_n(k_s a) - \beta_1 J_n(k_s a)]}{J_n(k_s a)\,N_n'(k_s a)\,(A^2 + B^2)} \times [A(\beta_2 J_n'(k_s a) - \alpha_2 N_n'(k_s a)) + B(\delta_2 J_n'(k_s a) - \gamma_2 N_n'(k_s a))] + 1\right\}.$$

The expressions for the radiation intensities and transverse forces of the I-field can be obtained from Eqs. (9) and (10) if the following substitutions are made:

$$\rho v \to I; \quad (1 - 1/\varepsilon_s\mu_s\beta^2) \to 1,$$

$$J_n(x) \to J_n'(x); \quad N_n(x) \to N_n'(x).$$

We may note that these formulas apply at any velocity. It is easy to show that when $v \to 0$, $W_i^{\rho I} \to 0$ while $F_{ri}^{\rho I}$ becomes the expression which describes the interaction between a charged filament or a conductor and a ferrite, which is well known in electrostatics and magnetostatics. Thus, in case II (Fig. 2, II) when $v \to 0$, we obtain from Eq. (9) the force on a unit length of the charged filament in a medium characterized by ϵ_2 exerted by a dielectric cylinder characterized by ϵ_1:

$$F_{r,2}^\rho = 2\rho^2(\varepsilon_2 - \varepsilon_1)\,a_1^2/\varepsilon_2(\varepsilon_1 + \varepsilon_2)\,a\,(a^2 - a_1^2), \quad (11)$$

from Eq. (10) we find the force which exerted on a conductor in a medium characterized by μ_2 by a

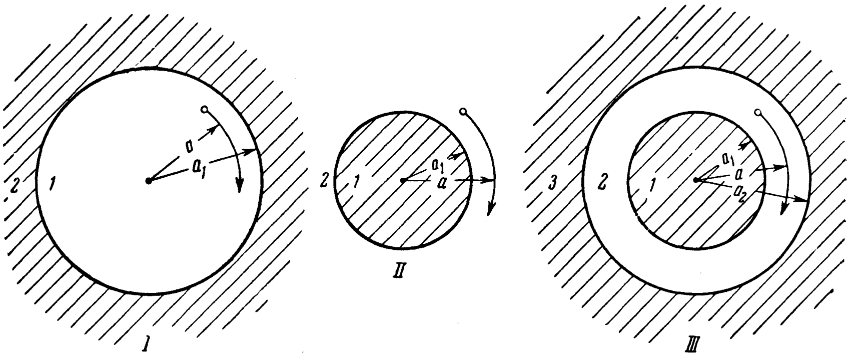

FIG. 2

cylinder characterized by a magnetic permeability μ_1:

$$F^{I}_{r,2} = 2I^2(\mu_1-\mu_2)\,a_1^2/\mu_2 c^2(\mu_1+\mu_2)\,a\,(a^2-a_1^2) \qquad (12)$$

(cf. for example, reference 8).

More interesting results are obtained for jets which move with relativistic velocities.

We consider jets which move in empty space $\epsilon_s = \mu_s = 1$) close to dense ferrite media (cases I and II), or inside a circular channel which is narrow compared with the radius of curvature, (case III) with velocities close to the velocity of light in vacuo $\{|(a_{1,2}-a)/a| = \Delta a/a;\ (\Delta a/a)^2 \ll 1\}$. The radiation intensity and the transverse currents are given by Bessel functions $J_n(x)$ and Neumann functions $N_n(x)$ and their derivatives for which the indices and arguments are approximately the same ($x \sim n$) or for which the arguments are considerably greater than the indices. The asymptotic formulas, which yield good approximations for these values of the indices and arguments, can be used to investigate the basic features of these cases (the asymptotic formulas for $J_n(x)$ and $N_n(x)$ when $x \sim n$ are given in references 5 and 6). We consider certain of these cases.

(a) If the jet moves in the azimuthal direction inside an empty circular channel which is narrow compared with the radius of curvature (cf. Fig. 2, III) and if $\epsilon_1 = \epsilon_3 = \epsilon$; $\mu_1 = \mu_3 = \mu$,

$$\varphi^{\rho}_{n,3} = \frac{J'^2_n(\sqrt{\varepsilon\mu}\,ka_1)}{J'^2_n(ka)}\,\frac{a_1\mu}{a_2}, \quad \text{if } \left(\frac{\lambda}{2\Delta a}\right) \gg \sqrt{\frac{2\Delta a}{a}}\left[\frac{\Gamma(2/3)}{\Gamma(4/3)}\right]^{3/2}, \qquad (13)$$

$$\varphi^{\rho}_{n,3} = \frac{2a}{a_2}\left\{\left(\varkappa+\frac{1}{\varkappa}\right)-\left(\varkappa-\frac{1}{\varkappa}\right)\sin\left[\frac{2n}{3}\left(\frac{2\Delta a}{a}\right)^{3/2}\right]\right\}^{-1}, \quad \text{if } \left(\frac{\lambda}{2\Delta a}\right) \ll \frac{4\Delta a}{a}, \qquad (14)$$

where we have introduced the notation $\kappa = \sqrt{2\epsilon\Delta a/a\mu}$ and λ is the wave length associated with the radiation of the jet.

Whence it follows that if the conditions in (13) are fulfilled, the expression for $W^{\rho}_{n,3}$ coincides with the expression for the radiation intensity of the n-th harmonic of the ρ-field of a jet which moves in a homogeneous space filled by a medium which surrounds the channel. The condition in (13) is analogous to the Ginzburg-Frank inequality[9] for a jet that moves in vacuum inside a circular region of space which is surrounded by a ferrite. It is easy to show that when (14) is satisfied the energy spectrum (maxima and minima on the curve which characterizes the spectral distribution of radiation losses) is such that the total intensity of the radiation beyond a point which corresponds to the frequency interval $\Delta\omega = (3\pi v/a)(a/2\Delta a)^{3/2}$ is equal to the radiation intensity of a charged jet that moves in vacuum [cf. Eq. (5)].

(b) In this case the transverse forces for the harmonics of the ρ-field which satisfy the condition in (14) are written in the following form:

$$F^{\rho}_{r,n,3} = \frac{2\pi k\rho^2(1-\beta^2)\,J_n(ka)\,J'_n(ka)\,(\varkappa-1/\varkappa)\cos[(2n/3)(2\Delta a/a)^{3/2}]}{(\varkappa+1/\varkappa)-(\varkappa-1/\varkappa)\sin[(2n/3)(2\Delta a/a)^{3/2}]}. \qquad (15)$$

It is apparent from (15) that the transverse force of the ρ-field depends on the parameter κ. If $\kappa > 1$, the harmonics of the field for which $n < n_{cr}$ (critical harmonics are those which satisfy the condition: $\frac{2}{3} n_{cr} (2\Delta a/a)^{3/2} = 2m \pm 1/2)\pi$, $m = 0, 1, 2, \ldots\ldots$ weaken the attractive force between the jet and the inner cylinder whereas the harmonics of the field for which $n > n_{cr}$ intensify this force. On the other hand, if $\kappa < 1$, the harmonics for which $n > n_{cr}$ weaken the attractive force whereas the harmonics below the critical harmonic intensify it.

(c) The functions $\phi^{I}_{n,3}$ and $F^{I}_{r,n,3}$ which characterize the I-field are obtained from Eqs. (13), (14) and (15) by means of the following substitutions: $\rho v \rightarrow I$, $\kappa \rightarrow \sqrt{2\mu\Delta a/a\epsilon}$, $(1-\beta^2) \rightarrow -1$; thus, for the I-field the parameter which characterizes the maxima and minima on the spectral intensity curve and determines transverse forces is the quantity $\sqrt{2\mu\Delta a/a\epsilon}$.

(d) A comparison of cases I and III shows that there is no difference when (14) is satisfied.

[1]A. I. Morozov, Вестник МГУ (Bulletin, Moscow State Univ.) No. 1, 72 (1957) ; Dissertation, Moscow State Univ. 1957.

[2]M. I. Kaganov, Тр. физич. отделения физ.-мат. ф-та ХГУ (Trans. Phys., Div., Phys.-Math. Faculties, Khar'kov State Univ.) 4, 67 (1953).

[3]N. Bohr, The Passage of Atomic Particles Through Matter (Russ. Transl.) IIL, M. 1950.

[4]Ivanenko and Sokolov, Классическая теория поля (Classical Theory of Fields) GITTL, M.-L. 1951; A. A. Sokolov, Вестник МГУ (Bulletin, Moscow State Univ.) No. 4, 77 (1947).

[5]A. N. Didenko, Радиотехника и электроника (Radio Engineering and Electronics) (in press).

[6]V. A. Fock, Dokl. Akad. Nauk SSSR, 1, 97 (1934).

[7]N. P. Klepikov, Dissertation, Moscow State Univ. 1953.

[8]Landau and Lifshitz, Электродинамика сплошных сред (Electrodynamics of Continuous Media) GITTL, M. 1957.

[9]V. L. Ginzburg and I. M. Frank, Dokl. Akad. Nauk SSSR 56, 699 (1947).

Translated by H. Lashinsky
134