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PHENOMENOLOGICAL EQUATIONS OF STATISTICAL DYNAMICS OF AN INCOMPRESSIBLE TURBULENT LIQUID

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A closed system of differential equations, which describe statistically the turbulent flow of a liquid, can be obtained only by limiting the number of variables that characterize the motion of the liquid. The appropriate variables to take here are the mean velocity $U_i = \bar{u}_i$ at a given point, and the mean pressure $P = \bar{p}$, and also the (one-sided) second moments of the fluctuating velocity $R_{ij} = v_i v_j$, and the turbulent viscosity N .

The equations for U_i are obtained from the usual averaging of the equations of hydrodynamics:

$$\frac{dU_i}{dt} + \frac{\partial R_{ih}}{\partial x_h} + \frac{\partial P}{\partial x} = 0, \quad \frac{\partial U_k}{\partial x_k} = 0. \quad (1)$$

In the averaging of the equation for $v_i v_j$, obtained from the Navier-Stokes equation, there appear third moments of the fluctuating velocity $v_i v_j v_k$, and also moments that contain derivatives with respect to the coordinates. All these moments can be expressed phenomenologically, in first approximation, in terms of the variables enumerated above and their derivatives. The requirements of dimensionality, tensor invariance, and parity must be satisfied here. If tensor combinations are chosen and known experimental data¹ are followed, then a quasi-diffusion equation is obtained for R_{ij} :

$$\begin{aligned} & \frac{dR_{ij}}{dt} - \frac{\partial}{\partial x_k} \left[\frac{1}{3} N \left(\delta_{ij} \frac{\partial R}{\partial x_k} + \delta_{jk} \frac{\partial R}{\partial x_i} + \delta_{ki} \frac{\partial R}{\partial x_j} \right) \right. \\ & \quad \left. - \alpha R \left(\delta_{ij} \frac{\partial N}{\partial x_k} + \delta_{jk} \frac{\partial N}{\partial x_i} + \delta_{ki} \frac{\partial N}{\partial x_j} \right) \right] \\ & + \frac{\partial}{\partial x_i} \left(\frac{1}{3} N \frac{\partial R}{\partial x_j} - \frac{5\alpha}{2} R \frac{\partial N}{\partial x_j} \right) + \frac{\partial}{\partial x_j} \left(\frac{1}{3} N \frac{\partial R}{\partial x_i} - \frac{5\alpha}{2} R \frac{\partial N}{\partial x_i} \right) \\ & + R_{ik} \frac{\partial U_j}{\partial x_k} + R_{jk} \frac{\partial U_i}{\partial x_k} + \frac{\beta}{N} \left(R_{ih} R_{jh} - \frac{\delta_{ij}}{3} R_{hl}^2 \right) + \frac{\delta_{ij}}{3} \frac{\gamma R^2}{N} = 0. \end{aligned} \quad (2)$$

Here $R = R_{kk}$. The physical meaning of the separate terms is as follows: the second term is connected with the transfer of turbulence energy ($v_i v_j v_k$ is contained in the square brackets); this transfer is directed, as is well known, from the walls to the middle of the flow. The maximum of R on the boundary between the turbulent boundary layer and the laminar underlayer serves as the source. The third and fourth terms take into account the transfer of energy by the fluctuating pressure; it proceeds in the opposite direction and partially cancels the second term. In summation, all these terms give:

$$\begin{aligned} & - \frac{\delta_{ij}}{3} \frac{\partial}{\partial x_k} \left(N \frac{\partial R}{\partial x_k} \right) + \alpha \delta_{ij} \frac{\partial}{\partial x_k} \left(R \frac{\partial N}{\partial x_k} \right) \\ & - \frac{3\alpha}{2} \left[\frac{\partial}{\partial x_i} \left(R \frac{\partial N}{\partial x_j} \right) + \frac{\partial}{\partial x_j} \left(R \frac{\partial N}{\partial x_i} \right) \right]. \end{aligned}$$

Further, the increase of the turbulent energy derives from the gradient of the mean velocity; it is obtained directly from the Navier-Stokes equation. The next-to-last term of the equation describes the shift of the anisotropy of the turbulence due to the scattering of the fluctuating velocity on the fluctuating pressure; finally, the last term represents the viscous dissipation of energy in fine-grained turbulence. The dimensional universal constants α , β , and γ must be determined by experiment.

With regard to the turbulent viscosity, in first approximation, for each given flow, we must set $N = \text{const}$ everywhere, with the exception of the boundary layer. In the turbulent boundary layer, N falls off with approach to the wall. In this case, however, there is an added condition: the divergence of the total flow of turbulent energy is practically equal to zero:

$$\frac{\partial}{\partial x_k} \left(N \frac{\partial R}{\partial x_k} \right) = 0. \quad (3)$$

This also gives the lacking equation for the determination of the value of N in the boundary layer.

As a condition for joining the solution of the equation for the two regions [the middle part of the flow ($N = \text{const}$) and the boundary layer — condition (3)] we use the requirement of continuity of all functions and their first derivatives. On the

walls, then, $U_i = 0$ and $N = 0$, and also $NR_{ij} = 0$.

Equations (1) to (3) are found to be in satisfactory quantitative agreement with experiment.¹ In this case the following approximate values are obtained for the constants: $\alpha = 5$; $\beta = 0.16$; $\gamma = 0.022$.

The approximate solution for the boundary layer can be obtained in the form of an expansion in powers of R_{yy}/R (x is directed along the current and y along the normal to the wall). The first approximation gives the following dependencies:

$$R_{xy} \frac{dU_x}{dy} \sim \frac{1}{y}, \quad R^3 \sim \log \frac{l}{y}, \quad N \sim y \left(\log \frac{l}{y} \right)^{2/3}, \quad (4)$$

$$R_{xy} \sim a - y, \quad R_{yy} \sim R_{xy}^2/R, \quad R_{zz} \sim R.$$

For flow in the tube, $2l = a$, where a is the radius of the tube.

These dependencies are confirmed by experiment. In particular, the logarithmic dependence for R^3 is confirmed.

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ON THE DEVIATION OF THE EQUILIBRIUM SHAPE OF ATOMIC NUCLEI FROM AXIAL SYMMETRY

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AN analysis of the experimental data on the levels of atomic nuclei indicates that a number of nuclei do not have an axis of symmetry.¹ Therefore it appears advisable to investigate the behavior of nucleons in a potential field without axial symmetry. As an example of such a field we consider an infinitely deep potential well with vertical walls which has the shape of an ellipsoid with semi-axes $a_x R_0$, $a_y R_0$, and $a_z R_0$, where R_0 is the radius of a sphere of equal volume. The problem of determining the nucleon states in such a potential well is reduced to the solution of the equation

$$(2M)^{-1} (\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2) \psi_i(\mathbf{r}) = E_i \psi_i(\mathbf{r}) \quad (1)$$

inside the ellipsoid with zero boundary conditions. By considering the deviations from spherical shape as being small we set

$$a_x^{-1} = 1 + \lambda, \quad a_y^{-1} = 1 + \nu, \quad a_z^{-1} = 1 + \kappa,$$

where κ is related to λ and ν by the condition that the volume remains the same. By going over in (1) to the variables

$$x' = x a_x^{-1}, \quad y' = y a_y^{-1}, \quad z' = z a_z^{-1}$$

and by restricting ourselves to quantities of the second order of smallness, we obtain

$$[(2M)^{-1} \hat{p}^2 + V(\mathbf{r}')] \psi_i(\mathbf{r}') = E_i \psi_i(\mathbf{r}'),$$

$$\psi_i(\mathbf{r}') = 0 \quad \text{for } r' = R_0,$$

$$V(\mathbf{r}') = (2M)^{-1} [(\alpha^2 + 1/3 \delta^2) \hat{p}^2$$

$$+ (\alpha - 3/4 \alpha^2 - 1/12 \delta^2) (\hat{p}_x^2 - 3 \hat{p}_z^2) + \delta (1 + 1/2 \alpha) (\hat{p}_x^2 - \hat{p}_y^2)],$$

$$\alpha = \lambda + \nu, \quad \delta = \lambda - \nu.$$

We regard $V(\mathbf{r}')$ as a perturbation. The wave functions of the unperturbed problem (up to the normalization factor) and the corresponding eigenvalues are equal to

$$\psi_i(\mathbf{r}') \equiv \psi_{nlm}(\mathbf{r}') = r'^{-1/2} j_{l+1/2}(\mu_{nl+1/2} r'/R_0) Y_{lm}(\theta', \varphi'),$$

$$E_i^0 \equiv E_{nl}^0 = \hbar^2 \mu_{nl+1/2}^2 / 2MR_0^2,$$

where $\mu_{nl+1/2}$ is the n -th root of the Bessel function $j_{l+1/2}(z)$, and Y_{lm} is a spherical harmonic.

Already in the first order of perturbation theory the degeneracy with respect to m is completely removed. Calculations carried out up to second-order perturbation theory inclusive lead to quite awkward expressions for the energies, which in the case $\delta = 0$ (axial symmetry) reduce to the corresponding expressions given by Moszkowski² (when $\delta = 0$, his parameter d is related to the parameter α by the expression $d = \alpha + \alpha^2/4$). Qualitatively the behavior of the nucleon levels can be studied as with the s and p shells as examples. In the case of s nucleons we obtain

$$E_{n0} = E_{n0}^0 \left\{ 1 + \left[1 - \frac{144}{5} s_2(\mu_{n+1/2}) \right] \left(\alpha^2 + \frac{1}{3} \delta^2 \right) \right\}. \quad (2)$$

and the p level splits up into three:

$$(E_{n1})_1 = E_{n1}^0 \left\{ 1 - \frac{4}{5} \alpha + \alpha^2 \left[\frac{8}{5} - \frac{432}{7} s_3(\mu_{n+1/2}) \right] + \delta^2 \left[\frac{2}{5} - \frac{80}{7} s_3(\mu_{n+1/2}) \right] \right\},$$

$$(E_{n1})_2 = E_{n1}^0 \left\{ 1 + \frac{2}{5} \alpha - \frac{2}{5} \delta + \alpha^2 \left[\frac{7}{10} - \frac{288}{7} s_3(\mu_{n+1/2}) \right] + \delta^2 \left[\frac{3}{10} - \frac{128}{7} s_3(\mu_{n+1/2}) \right] - \alpha \delta \left[\frac{1}{5} - \frac{96}{7} s_3(\mu_{n+1/2}) \right] \right\},$$

$$(E_{n1})_3 = E_{n1}^0 \left\{ 1 + \frac{2}{5} \alpha + \frac{2}{5} \delta + \alpha^2 \left[\frac{7}{10} - \frac{288}{7} s_3(\mu_{n+1/2}) \right] + \delta^2 \left[\frac{3}{10} - \frac{128}{7} s_3(\mu_{n+1/2}) \right] + \alpha \delta \left[\frac{1}{5} - \frac{96}{7} s_3(\mu_{n+1/2}) \right] \right\},$$