GENERALIZED SELF-CONSISTENT FIELD EQUATIONS

P. S. ZYRIANOV

Ural' Polytechnic Institute

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The Hartree equations generalized to the case of nonstationary states are modified to incorporate processes of scattering of particles at small distances as a sort of momentum interaction. The relation of these equations to the theories of Landau and Silin is shown. On the basis of the generalized equations, dispersion relations are found for Fermi and Bose systems of interacting particles.

In systems of interacting particles, the density fluctuations can be divided conditionally into "coarse-grained" and "fine-grained" fluctuations.[1] This separation is related to the approximate methods of description of the sole actually occurring fluctuations. In condensed systems, the "coarse-grained" fluctuations describe density changes over regions of space with linear dimensions which are appreciably greater than the distance between particles, while the changes in number density of particles over space regions of dimensions comparable with or smaller than the average distance between particles are described by the "fine-grained" fluctuations. The "coarse-grained" fluctuations, in which the properties of the system manifest themselves as if it were a continuous medium, can be described by means of the Hartree self-consistent field. This field contains the effect of long range force correlations in the system.

The self-consistent field is too poor an approximation to describe the "fine-grained" fluctuations. This is due primarily to the large fluctuations of the actual (but not of the average) field at small distances, even for the case of a spatially uniform density. Processes of interaction of particles at small distances have rather the character of collisions, and they are to be described preferably in terms of a scattering problem. Here the fundamental quantity is the scattering amplitude, which depends on the potential of the interaction between the particles and on their momenta. If we neglect the effect of the self-consistent Hartree field on the scattering processes at small distances, the scattering amplitude will be independent of the spatial coordinates. This statement is equivalent to neglecting the interaction between the "coarse-grained" and "fine-grained" fluctuations.

The interaction energy of the particles at small distances can then be assumed to be some function of the scattering amplitude. In the simplest case of small scattering amplitudes, this function can be expanded in series and one can limit oneself to the linear approximation; the interaction energy will then be proportional to the scattering amplitude. In general the interaction at small distances can be regarded as a sort of momentum interaction, and the scattering amplitude or some function of it as the operator of this interaction.

As will be clear from the sequel, such interactions describe the force correlations over small distances. The momentum interactions described above can be included in the Hartree equations generalized to the case of nonstationary states,[2] and can be expressed in the following form:

$$i\hbar\dot{\psi}_j = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \mathbf{q}^2} + \int G(|\mathbf{q}-\mathbf{x}'|)\sum_i |\psi_i(\mathbf{x}')|^2 d\mathbf{x}'\right\}\psi_j(\mathbf{q}) + \sum_i \int \psi_i(\mathbf{x}')\,\Phi(\hat{\mathbf{p}}, \hat{\mathbf{p}}'^*; \mathbf{q}, \mathbf{x}')\,\psi_i^*(\mathbf{x}')\,d\mathbf{x}'\psi_j(\mathbf{q}), \tag{1}$$

where

$$\Phi(\hat{\mathbf{p}}, \hat{\mathbf{p}}'^*; \mathbf{q}, \mathbf{x}') = {}^1/_2[\delta(\mathbf{x}'-\mathbf{q})F(|\hat{\mathbf{p}}-\hat{\mathbf{p}}'^*|) + F(|\hat{\mathbf{p}}-\hat{\mathbf{p}}'^*|)\delta(\mathbf{x}'-\mathbf{q})],$$

$$\hat{\mathbf{p}} = -i\hbar\partial/\partial\mathbf{q}, \quad \hat{\mathbf{p}}'^* = i\hbar\partial/\partial\mathbf{x}',$$

$F(|\hat{\mathbf{p}}|)$ is some function of the scattering amplitude.

One can establish the relation of these equations to the Landau equation for a Fermi liquid[3] and with the equation for a degenerate electron fluid which was obtained by Silin[1] as a generalization of the work of Landau, as well as with the Hartree-Fock equations.

A more complete and consistent description of fluctuations, not assuming a separation of the fluc-

tuations into "coarse and fine grained", should start from a single interaction kernel depending on coordinates as well as momenta of the interacting particles. In Eq. (1) the interaction kernel is a sum of coordinate and momentum interactions. This is the cause of the formal division of the fluctuations into two types.

1. To establish the connection of (1) with the Landau and Silin equations, which arise from (1) as a quasi-classical approximation in the limit $\hbar \to 0$, we introduce the density matrix

$$\rho(\mathbf{q}, \mathbf{q}') = \sum_i \psi_i(\mathbf{q})\, \psi_i^*(\mathbf{q}'). \tag{2}$$

For the case of particles with spin, the spin coordinates may be assumed to be contained in $\mathbf{q}$.

Using (1), we can find the equation of motion for $\rho(\mathbf{q}, \mathbf{q}')$. This equation has the following form:

$$\begin{aligned} i\hbar\dot{\rho}(\mathbf{q}, \mathbf{q}') = \Big\{ &-(\hbar^2/2m)(\partial^2/\partial\mathbf{q}^2 - \partial^2/\partial\mathbf{q}'^2) \\ &+ \int dx' [G(|\mathbf{q} - \mathbf{x}'|) \\ &- G(|\mathbf{q}' - \mathbf{x}'|)]\rho(\mathbf{x}', \mathbf{x}') \\ &+ \int dx''\, dx' \delta(\mathbf{x}' - \mathbf{x}'')\, \Phi(\mathbf{q}, \mathbf{x}'; \hat{\mathbf{p}}, \hat{\mathbf{p}}'^*)\rho(\mathbf{x}'', \mathbf{x}') \\ &- \int dx''\, dx' \delta(\mathbf{x}'' - \mathbf{x}')\, \Phi(\mathbf{q}', \mathbf{x}'; \hat{\mathbf{p}}, \hat{\mathbf{p}}'^*)\rho(\mathbf{x}', \mathbf{x}'')\Big\}\rho(\mathbf{q}, \mathbf{q}'). \end{aligned} \tag{3}$$

We can go over from this equation to the equation of motion for the quantum distribution function $f(\mathbf{p}, \mathbf{q})$, which is related to $\rho(\mathbf{q}, \mathbf{q}')$ by the well known relation[5]

$$f_{s,s'}(\mathbf{p}, \mathbf{q}) = \frac{1}{(2\pi)^3}\int \rho(\mathbf{q} - \hbar\boldsymbol{\tau}/2;\ \mathbf{q} + \hbar\boldsymbol{\tau}/2)\, e^{-i\boldsymbol{\tau}\rho}\, d\tau, \tag{4}$$

where s and s' are spin indices. We can introduce a distribution function $f(\mathbf{p}, \mathbf{q})$ which is independent of the spin indices:

$$f(\mathbf{p}, \mathbf{q}) = \sum_s f_{ss}(\mathbf{p}, \mathbf{q}). \tag{5}$$

If we assume that the magnetization of the system is equal to zero and neglect fluctuations of the spin density in phase space, we can obtain the following quantum-kinetic equation for $f(\mathbf{p}, \mathbf{q})$ by using Eq. (3):

$$\begin{aligned} &\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial f}{\partial \mathbf{q}} - \frac{i}{(2\pi)^3\hbar}\int dx'\, dp'\, dp''\, d\tau [G(|\mathbf{x}' - \mathbf{q} + \hbar\boldsymbol{\tau}/2|) \\ &- G(|\mathbf{x}' - \mathbf{q} - \hbar\boldsymbol{\tau}/2|)]\, f(\mathbf{p}', \mathbf{x}')\, f(\mathbf{p}'', \mathbf{q}) \exp\{i\boldsymbol{\tau}(\mathbf{p}'' - \mathbf{p})\} \\ &- \frac{i}{(2\pi)^3\hbar}\int d\varkappa\, d\tau\, dp'\, dp''\, F(|\varkappa|)\{f(\mathbf{p}', \mathbf{q} + (\hbar\boldsymbol{\tau}/2) - \hbar\boldsymbol{\varkappa}/2) \\ &\times f(\mathbf{p}'', \mathbf{q} - \hbar\boldsymbol{\varkappa}/2) - f(\mathbf{p}', \mathbf{q} - (\hbar\boldsymbol{\tau}/2) + h\boldsymbol{\varkappa}/2) \\ &\times f(\mathbf{p}'', \mathbf{q} + \hbar\boldsymbol{\varkappa}/2)\} \exp\{i\boldsymbol{\varkappa}(\mathbf{p}'' - \mathbf{p}') + i\boldsymbol{\tau}(\mathbf{p}'' - \mathbf{p})\}. \end{aligned} \tag{6}$$

Making the limiting transition $\hbar \to 0$ in (6), we obtain the quasi-classical kinetic equation in the following form:

$$\begin{aligned} &\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{q}}(\mathbf{v}, f) + \frac{\partial}{\partial p}(\mathbf{P}, f) = 0, \\ &\mathbf{v} = \partial\varepsilon/\partial\mathbf{p}, \quad \mathbf{P} = -\partial\varepsilon/\partial\mathbf{q}; \end{aligned} \tag{7}$$

$$\begin{aligned} \varepsilon(\mathbf{p}, \mathbf{q}) = \frac{p^2}{2m} &+ \int F(|\mathbf{p} - \mathbf{p}'|)\frac{\partial f(\mathbf{p}'\mathbf{q})}{\partial \mathbf{p}'}\, dp' \\ &+ \int G(|\mathbf{q} - \mathbf{x}'|) f(\mathbf{p}', \mathbf{x}')\, dx'\, dp'. \end{aligned} \tag{8}$$

If we neglect the last term in (8), which describes the energy of interaction of the particle with the Hartree field, we get a kinetic equation analogous to the corresponding equation in reference 3.

The function $\epsilon(\mathbf{p}, \mathbf{q})$ is the energy of the particle in the self-consistent Hartree field, including scattering processes at small distances which describe force correlations of small range; $\mathbf{v}$ and $\mathbf{P}$ can be treated as the velocity and the self-consistent force, respectively.

We note that Eq. (7), though it is a quasi-classical approximation, contains exchange effects or, in other words, takes account of the symmetry of the wave function of the initial system. Actually the initial equations (1), from which (7) was obtained, go over into the Hartree-Fock equations if we set

$$F(|\mathbf{p}|) = \int G(|\mathbf{r}|)\, e^{i\mathbf{r}\cdot\mathbf{p}/\hbar}\, dr, \tag{8a}$$

where $G(|\mathbf{r}|)$ is the interaction potential between the particles in vacuum.

2. To find the energy spectrum of low excited states of a system of interacting particles, it is preferable to use Eq. (1) rather than (7). The point is that (1) contains quantum effects which are essential for Bose systems near the ground state (at absolute zero), while the equations of (7) no longer contain them since they are obtained in the limit $\hbar \to 0$.

A procedure for finding dispersion equations for the states of a system which are close to a homogeneous, isotropic distribution of particles in coordinate space and chaotic in the velocities was given briefly in reference 2. We shall follow this procedure.

Let us represent the function $\psi_j(\mathbf{x})$ as

$$\psi_j(\mathbf{x}, t) = \rho_j^{1/2}(\mathbf{x}, t) \exp\{(i/\hbar) S_j(\mathbf{x}, t)\}. \tag{9}$$

Substituting (9) in (1), we get a system of equations for the functions ρ_j, S_j. Since the state of the system with constant density, $\rho_j = \text{const}$, is an exact solution, the nonlinear system of equations for the functions ρ_j and S_j can be linearized in the neighborhood of this exact solution with $\rho_j = \text{const}$.

If we look for a solution of the linear equations

for ρ_j and S_j in the form of a superposition of plane traveling waves with frequency ω and wave vector $\mathbf{k}$, we find the following dispersion equation for Bose systems:

$$\omega^2 = (k^2/m^*)\langle N_0 G^*(k) + \hbar^2 k^2/4m^*\rangle; \qquad (10)$$

Here $G^*(k) = G(k) + \frac{1}{2}[F(\hbar k) + F(0)]$, $G(k)$ is the Fourier component of the interaction kernel, $F(x)$ is the kernel of the momentum interaction, and m^* is the effective mass which is defined by the expression:

$$(m^*)^{-1} = (1/m)\langle 1 + (3m/\hbar^2 k^2)[F(\hbar k) - F(0)]\rangle.$$

The structure of the energy spectrum (10) of the weakly excited states of a Bose system coincides in form with the results which were first obtained by Bogoliubov,[6] but differs in that the mass of the original particles is replaced by their effective mass, and that the function $\frac{1}{2}[F(\hbar k) + F(0)]$, which takes account of correlations over small distances, is added to the Fourier component of the kernel of the interaction between particles. If we define $F(x)$ from (8a), Eq. (10) coincides with the corresponding result from the Hartree-Fock equation.

In the case of perfectly elastic spheres of diameter a, the interaction kernel can be taken in the form

$$G(|\mathbf{q} - \mathbf{x}'|) = (\hbar^2/ma^2)\,\delta(|\mathbf{q} - \mathbf{x}'| a). \qquad (11)$$

Then Eq. (10) gives results which are analogous to those of Brueckner and Sawada[7] (cf. also reference 8), and in the limit $k \to 0$ with those of Lee and Yang.

In the case of systems of Fermi particles, to find the spectrum of collective oscillations we get the dispersion equation

$$1 = \sum_j (k^2/m_j^*)\, G_j^*(k)\langle[\omega - \mathbf{k}\mathbf{v}_j^*]^2 - \hbar^2 k^4/4m_j^*\rangle^{-1}, \qquad (12)$$

where

$$G_j^*(k) = G(k) + {}^1/_4 \sum_i [F(|\mathbf{p}_i - \mathbf{p}_j + \hbar\mathbf{k}|) - F(|\mathbf{p}_i - \mathbf{p}_j - \hbar\mathbf{k}|) + 2F(|\mathbf{p}_i - \mathbf{p}_j|)],$$

$$(m_j^*)^{-1} = (1/m) + (1/\hbar^2 k^2)\sum_i [F(|\mathbf{p}_i - \mathbf{p}_j + \hbar\mathbf{k}|) + F(|\mathbf{p}_i - \mathbf{p}_j - \hbar\mathbf{k}|) - 2F(|\mathbf{p}_j - \mathbf{p}_i|)],$$

$$\mathbf{k}\mathbf{v}_j^* = (\mathbf{k}\mathbf{p}_j/m) + (1/2\hbar)\sum_i [F(|\mathbf{p}_i - \mathbf{p}_j + \hbar\mathbf{k}|) - F(|\mathbf{p}_i - \mathbf{p}_j - \hbar\mathbf{k}|)],$$

and $\mathbf{p}_j$ is the momentum of the j-th particle in the state with constant density.

In the case of Coulomb forces, as $k \to 0$ we get the following expression for the limiting frequency of plasma oscillations:

$$\omega_0^2 = \frac{4\pi N_0 e^2}{m^*}; \quad \frac{1}{m^*} = \frac{1}{m} + \sum_{ij} \frac{2}{N_0} \frac{\partial^2 F(|\mathbf{p}_i - \mathbf{p}_j|)}{\partial \mathbf{p}_i^2}.$$

If we define $F(|\mathbf{p}|)$ by Eq. (8a), the dispersion equation (12) coincides with the corresponding equation obtained from the Hartree-Fock equations.

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