DISPERSION RELATIONS IN NON-RELATIVISTIC SCATTERING THEORY

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It is proposed that the dispersion relations for the scattering amplitude for potential scattering be deduced by studying the Green's function of the total Hamiltonian. The method is illustrated for the case of a three-dimensional non-relativistic equation for scattering from a fixed center. The relations previously obtained by Wong and Khuri are derived rigorously. The existence of dispersion relations is related to the problem of a complete description of the S matrix for various potential scattering equations.

SEVERAL recent papers appeared again to direct attention to the properties of the S matrix elements for complex energies in the non-relativistic theory.[1,2] It seems that, contrary to the previous work of van Kampen,[3] it is not necessary to require the validity of any special principle of causality. The locality of the potential in position space guarantees all properties necessary for a derivation of the dispersion relations.

The dispersion relations are derived in the paper of Wong[1] under the assumption that the perturbation expansion converges. This is too much of a restriction. The derivation of Khuri is based on the Fredholm theory for the integral equation of the scattering theory developed by Jost and Pais.[4] This involves a rather unwieldy mathematical apparatus; furthermore, the paper of Jost and Pais contains an incorrect proof of the assertions used by Khuri.

We propose a comparatively simple derivation of the dispersion relations based on the properties of the Green's function of the total Hamiltonian. The derivation is general and applies to an arbitrary potential scattering equation if the behavior of the corresponding Green's function is known. Our discussion uses the example of scattering from a three-dimensional center. The corresponding Green's function was investigated by Povzner.[5,6] The asymptotic behavior of this function for high energies was investigated by the author.[7,8]

The second part of the paper contains some considerations of mathematical character, which may be useful in discussing the possibilities for applying the dispersion relations.

1. DERIVATION OF THE DISPERSION RELATIONS FOR THE SCATTERING FROM A THREE-DIMENSIONAL CENTER

As is known, for the existence of a dispersion relation of the non-relativistic type

$$\mathrm{Re}\, f(E) = \frac{1}{\pi} \mathrm{P} \int_0^\infty \frac{\mathrm{Im}\, f(E')}{E' - E}\, dE' + \sum_j \frac{d_j}{E - E_j}, \qquad (1)$$

$$0 \leqslant E < \infty,$$

for the amplitude f(E), the following properties of f(E) are sufficient:

(a) f(E) is the limiting value on the positive real axis of a function which is analytic in the plane of the complex variable E with a cut along the positive real axis, except for a finite number of points E_j on the negative real axis, where the function may have simple poles with real residues d_j;

(b) f(E + i0) = f*(E − i0);

(c) for large |E|, f(E) → 0.

We consider the matrix element of the T matrix

$$T_{ab} = \langle \Phi_a | V | \Psi_b^{(+)} \rangle, \quad E_a = E_b. \qquad (2)$$

Here Φ_a is an eigenfunction of the free Hamiltonian H_0, and $\Psi_a^{(+)}$ is a solution of the integral equation of the scattering theory

$$\Psi_a^{(+)} = \Phi_a - G_0(E_a + i0) V \Psi_a^{(+)}. \qquad (3)$$

Here $G_0(E)$ is the Green's function of the free Hamiltonian.

Instead of Eq. (3) we shall use the representation of the function $\Psi_a^{(+)}$ in terms of the Green's function of the total Hamiltonian $H = H_0 + V$:

$$\Psi_a^{(+)} = \Phi_a - \lim_{\varepsilon\to 0} G(E_a + i\varepsilon) V\Phi_a. \quad (4)$$

For complex energies, G(E) is the solution of the equation

$$G(E) = G_0(E) - G_0(E) VG(E). \quad (5)$$

This expression we now use in writing down the elements of the T matrix:

$$T_{ab} = \langle\Phi_a | V | \Phi_b | - \lim_{\varepsilon\to 0} \langle\Phi_a | VG(E_a + i\varepsilon) V | \Phi_b\rangle. \quad (6)$$

For scattering from a three-dimensional center, where

$$H_0 = -\nabla^2, \quad V = \delta^3(\mathbf{x} - \mathbf{x}') V(\mathbf{x}),$$

we have:

$$\Phi_a = \frac{E_a^{1/4}}{\sqrt{2}} \left(\frac{1}{2\pi}\right)^{3/2} \exp\{i\sqrt{E_a}(\mathbf{x}, \alpha)\},$$
$$(\alpha, \alpha) = 1; \; G_0(E) = \frac{\exp\{i\sqrt{E}\,|\mathbf{x} - \mathbf{x}'|\}}{4\pi\,|\mathbf{x} - \mathbf{x}'|} \quad (7)$$

The functions $\Psi_a^{(+)}$ depend on the same parameters as Φ_a, i.e.,

$$\Psi_a^{(+)} = \Psi(\mathbf{x};\ E_a,\ \alpha). \quad (8)$$

We consider the resolvent G(E) = G(**x**, **x'**, E). In the following we shall assume that the potential satisfies the condition

$$|V(\mathbf{x})| \leqslant C/|\mathbf{x}|^{3+\varepsilon}, \quad \varepsilon > 0. \quad (9)$$

Povzner, investigating the integral equation (5), obtained the following properties* of the function G(**x**, **x'**, E):

(1) The function G(**x**, **x'**, E) is analytic in the complex plane of E with a cut along the positive real axis, except for a finite number of points E_j on the negative real axis and, possibly at the origin, where it has simple poles. The points E_j correspond to discrete eigenvalues and

$$\operatorname{Res} G(\mathbf{x},\ \mathbf{x}',\ E)|_{E=E_j} = \sum_k \psi_k(\mathbf{x})\,\psi_k(\mathbf{x}'), \quad (10)$$

where $\psi_k(\mathbf{x})$ are orthonormal eigenfunctions corresponding to the eigenvalue E_j.

(2) The limit

$$\lim_{\varepsilon\to 0} G(\mathbf{x},\ \mathbf{x}',\ E + i\varepsilon), \quad \varepsilon > 0$$

exists for all E > 0, with the exclusion, perhaps, of a finite number of points $E_i > 0$, for which the homogeneous equation

$$u = G_0(E_i + i0) Vu \quad (11)$$

has a non-trivial solution. At these points G(**x**, **x'**, E) has a singularity of type

$$G(\mathbf{x},\ \mathbf{x}',\ E) \sim \sum u_k(\mathbf{x})\, u_k(\mathbf{x}')/(E - E_i),$$

where $u_k(\mathbf{x})$ are linearly independent solutions of equation (11). These solutions satisfy the requirement

$$\int \exp\{i\sqrt{E_i}\,(\mathbf{x},\ \alpha)\}\, V(\mathbf{x})\, u(\mathbf{x})\, d\mathbf{x} = 0 \quad (12)$$

for arbitrary α, $(\alpha, \alpha) = 1$.

We note here that the values E > 0 for which equation (11) has non-trivial solutions correspond to a positive discrete spectrum, the absence of which has so far not been rigorously proven for a potential satisfying only condition (9). It is precisely this point which has been treated incorrectly in the paper of Jost and Pais. From their considerations it follows only that the solutions of Eq. (11) satisfy condition (12), but it does not follow that u = 0.

(3) The following condition is satisfied:

$$G(\mathbf{x},\ \mathbf{x}',\ E + i0) = G^*(\mathbf{x},\ \mathbf{x}',\ E - i0). \quad (13)$$

The behavior of G(**x**, **x'**, E) for large |E| was investigated by the author.

(4) From the results of references 7 and 8 it follows that

$$\left| G(\mathbf{x},\ \mathbf{x}',\ E) - \frac{1}{4\pi} \frac{\exp\{i\sqrt{E}\,|\mathbf{x} - \mathbf{x}'|\}}{|\mathbf{x} - \mathbf{x}'|} \right| \leqslant C_E \times \exp\{-\operatorname{Im}\sqrt{E}\,|\mathbf{x} - \mathbf{x}'|\}, \quad (14)$$

where $C_E \to 0$ for $|E| \to \infty$.

The properties (1) to (4) are sufficient for the derivation of the dispersion relations. We consider the forward scattering amplitude:

$$T_{aa} = \left(\frac{1}{2\pi}\right)^3 \frac{\sqrt{E}}{2} \int V(\mathbf{x})\, d\mathbf{x} - \frac{\sqrt{E}}{2} \left(\frac{1}{2\pi}\right)^3 \frac{1}{4\pi} \int e^{i\sqrt{E}(\mathbf{x},\alpha)}\, V(\mathbf{x})\, G(\mathbf{x},\ \mathbf{x}',\ E)\, V(\mathbf{x}')\, e^{-i\sqrt{E}(\mathbf{x}',\alpha)} d\mathbf{x}\, d\mathbf{x}'. \quad (15)$$

Using the obvious estimate

$$|\exp\{i\sqrt{E}\,(\mathbf{x} - \mathbf{x}',\ \alpha)\}| \leqslant \exp\{\operatorname{Im}\sqrt{E}\,|\mathbf{x} - \mathbf{x}'|\}$$

and relation (14), we obtain the result that the integral on the right hand side of (15) is absolutely convergent for all E of the complex plane. Due to the analyticity of the function under the integral, the integral itself is analytic in the whole plane, excluding the cut and the points of singularity of the function G(E). At these points we have simple poles with the residues

*These results remain valid when the potential has singularities of type $\alpha/|\mathbf{r}|^{2-\varepsilon}$ at a finite distance. The integral condition of Khuri is equivalent to this requirement.

$$\gamma_j = \frac{i\sqrt{|E_j|}}{2}\, d_j = -\frac{i\sqrt{|E_j|}}{4}\left(\frac{1}{2\pi}\right)^4$$

$$\times \sum_k \int \exp\{\sqrt{|E_j|}\ (\mathrm{x},\ \alpha)\}\, V(x)\, \psi_k(\mathrm{x})\, d\mathrm{x}$$

$$\times \int \exp\{-\sqrt{|E_j|}(\mathrm{x}', \alpha)\}\, V(\mathrm{x}')\, \psi_k(\mathrm{x}')\, d\mathrm{x}'.$$

Since the eigenfunctions can be chosen to be real, the d_j are real.

There are no singularities on the cut due to property (12).

Thus, the function

$$f(E) = \frac{2}{\sqrt{E}}\left[T_{aa} - \frac{\sqrt{E}}{2}\left(\frac{1}{2\pi}\right)^3 \int V(\mathrm{x})\, d\mathrm{x}\right] \qquad (16)$$

has been shown to have the property (a). Furthermore, property (b) for $f(E)$ follows from property (13) for $G(\mathbf{x}, \mathbf{x}', E)$. Finally, one can show with the help of (14) that $f(E)$ also has the property (c). From this there follows a dispersion relation of type (1) for the function $f(E)$, which can also be written in terms of the forward scattering amplitude.

Similar considerations apply for the scattering amplitude with fixed transferred momentum τ. In this case condition (9) is not sufficient, and we must require that

$$|V(\mathrm{x})| \leqslant C|x|^{-3-\varepsilon}\exp(-\tau|\mathrm{x}|/2), \quad \varepsilon > 0$$

(cf. reference 2 on this point).

2. DISCUSSION OF THE DISPERSION RELATIONS

In the above derivation it was evident that we did not require any special causality conditions, except for the locality of the potential in position space, for the existence of dispersion relations in the case of scattering from a three-dimensional center. The locality of the potential apparently plays the same role in the non-relativistic theory as the requirement of microscopic causality in field theory.

We add a number of formal considerations, which may be useful for an analysis of the reasons for the existence of dispersion relations, and of the possibilities for applying them.

In non-relativistic scattering problems in which dispersion relations appear, the situation is such that the complete S matrix of the system is a function of a greater number of variables than the potential. The general requirements which the S matrix has to fulfil, such as unitarity and symmetry, do not sufficiently lower the number of variables. For example, for scattering from a three-dimensional center, the S matrix is a complex function of the energy E, $0 \leq E < \infty$, and of two unit vectors defining the directions of the incident and the scattered particles, while the potential is a real point function in three-dimensional space, i.e., a function of the radius r, $0 \leq r < \infty$, and of one unit vector.

The requirements of symmetry and unitarity lead to the possibility of expressing the S matrix elements through a real and symmetric function of the energy and two unit vectors. For example, one can use the representation of the S matrix in terms of the K matrix:

$$S = (1 + iK)(1 - iK)^{-1}. \qquad (17)$$

But this function still depends on a greater number of parameters than the potential. An analogous situation obtains even for the scattering from a one-dimensional potential barrier. In this case the S matrix is a 2×2 unitary and symmetric matrix

$$S(E) = \begin{pmatrix} a(E) & b(E) \\ b(E) & c(E) \end{pmatrix}, \qquad (18)$$

consisting of the coefficients for reflection to the right and to the left, $a(E)$ and $c(E)$, and the transmission coefficient, $b(E)$. The requirements of symmetry and unitarity lead to the possibility of expressing all S matrix elements by three real functions of E, $0 \leq E < \infty$, for example, by $a(E)$ and the argument of the function $b(E)$, while the potential is a real function of the variable x, $-\infty < x < \infty$, which may be regarded as two real functions of a variable going from 0 to ∞.

Since the potential determines the S matrix uniquely, it is natural to expect that in these cases there exist additional relations between the S matrix elements, which lower the number of variables on which these elements depend. The dispersion relations are examples for such connections. This is particularly clear in the example of scattering from a one-dimensional barrier. In this case an analysis similar to that of part 1 shows that the transmission coefficient $b(E)$ is the limiting value of a function which is analytic in the cut plane of E and has simple poles at points corresponding to a discrete spectrum. For large $|E|$

$$b(E) = 1 + O(|E|^{-1/2}), \qquad b(E + i0) = b^*(E - i0).$$

Thus, conditions (a), (b), and (c) are satisfied for the function $f(E) = b(E) - 1$, and thus there exists a relation of type (1). It is more convenient to consider the function $g(E) = \ln b(E)$. We obtain a relation for it which in terms of $b(E)$ has

the form:

$$\arg b(E) = \frac{1}{\pi} P \int \frac{\ln |b(E')|}{E' - E} dE' + 2 \sum_j \arctan \left(\frac{|E_j|}{E} \right)^{1/2}. \tag{19}$$

Relation (19) allows us to construct the whole S matrix from the reflection coefficient a(E). Indeed, from the unitarity requirement we have

$$|b(E)| = (1 - |a(E)|^2)^{1/2}; \qquad c^*(E) = -a(E) b^*(E) / b(E). \tag{20}$$

We determine arg b(E) from formula (19), and we obtain b(E) and c(E).

In the three-dimensional case the relation introduced for T_{aa} does not sufficiently reduce the number of parameters. There must exist a whole series of additional relations, possibly of a completely different type. Here it is interesting to note a result of Wong and Toll.[9] For a spherically symmetric potential they obtain the result that, together with the forward scattering amplitude

$$f_0 = \sum_{l=0}^{\infty} (2l + 1) [S_l(E) - 1] \tag{21}$$

the expressions

$$f_n(E) = \left(\frac{1}{E} \right)^n \sum_{l=n}^{\infty} (2l + 1) \frac{(l+n)!}{(l-n)!} [S_l(E) - 1] \tag{22}$$

have properties of types (a), (b), and (c). Hence there exists an infinite system of relations connecting all $S_l(E)$. Here it appears that only one function among all the $S_l(E)$ can be taken independently. Under the assumption that the perturbation theory converges, all $S_l(E)$ can then be expressed through $S_0(E)$.

The above examples show that if the potential depends on less parameters than the S matrix, then there exist a whole series of relations between its elements, which reduce the number of variables on which the S matrix depends. On the other hand, in the one-dimensional case, where the S matrix is a simple function with modulus one [$S(E) = \exp\{2i\delta(E)\}$, where $\delta(E)$ is a real function of E] and where the potential also depends on one variable going from 0 to ∞, relations of this type do not exist. Indeed, it follows from the theory of the reverse problem adapted to this case by Krein and Marchenko,[10,11] that for every function S(E) such that $|S(E)| = 1$, $S(0) = \pm 1$, and $S(E) - 1 \to 0$ for $E \to \infty$ sufficiently fast, one can find a potential, which is local in position space, and for which S(E) is the S function. Thus the S function corresponding to a potential which is not subject to any requirements except that of locality (and, of course, to the requirements which guarantee the existence of the S function, e.g., the decrease like $1/r^{2+\epsilon}$ for $r \to \infty$) cannot satisfy any relations of the dispersion type. We note here that our assertions do not contradict the results of van Kampen,[3] since the causality principle, which is the basis for his derivation of the dispersion relations, is equivalent to the assumption that the potential vanishes outside a finite interaction range. As is known, $\tilde{S}(E) = S(E) \exp(2iE^{1/2}a)$ satisfies the dispersion relation if $V(r) = 0$ for $r > a$.

Analogous considerations lead to the result that, in the case of scattering from a one-dimensional barrier, the relation (19) is the only relation, except for the requirements of symmetry and unitarity, to which the elements of the S matrix may be subject. Indeed, an investigation of the reverse problem, conducted for this case by Moses[12] and the author,[13] shows that for every function a(E) such that $|a(E)| < 1$, except at $E = 0$, where possibly $a(0) = -1$, and such that $a(E) \to 0$ for $E \to \infty$, one can find a potential for which the matrix constructed from a(E) with the help of formulas (19) and (20) is the S matrix.

Finally, the relations of Toll and Wong for the case of a three-dimensional spherically symmetric potential also are a complete system of relations in the sense that, if there exists a system of functions $S_l(E)$ subject to the conditions resulting from the analyticity properties (22), then it is possible to construct a spherically symmetric potential for which this system is a system of eigenvalues of the S matrix.

This last result shows that any relations between the S matrix elements which may later be introduced are consequences of the relations of Toll and Wong, although they may be different in form. This does not mean that the relations of Toll and Wong are the most convenient system. This system has a whole series of deficiencies. Firstly, one must require the fast decrease of the potential for $r \to \infty$ to guarantee the convergence of the series of type (22). Thus, for the convergence of the series determining f_n it is necessary that the integral $\int_0^{\infty} r^{2n+1} |V(r)| dr$ converge; thus, for f_n to have a meaning, the potential has to decrease faster than r^{-m} for arbitrary m. Interpreting the summation in (22) in some general sense, we can justify the relation of the Toll-Wong type also for an arbitrary potential (which decreases faster than $1/r^2$), but this has not yet been done. Secondly, it is not clear what the analogous relations in the spherically non-symmetric case look like. Therefore, one wishes to find pos-

sibly a new system of relations, which is valid also in the general case. In this connection it is interesting to note a paper by Moses,[14] in which a formal procedure is developed for the construction of a potential from the known backward scattering amplitude for all energies and all directions of the incident particles over a hemisphere. Moses points out the analogy between these data and the reflection coefficient in the case of a uni-dimensional barrier. It is of interest to note that these data consist of two real functions of the parameter E and a unit vector describing a hemisphere, while the potential depends on the parameter r and on a unit vector describing a full sphere.

[1] D. J. Wong, Phys. Rev. **107**, 302 (1957).
[2] N. N. Khuri, Phys. Rev. **107**, 1148 (1957).
[3] N. G. van Kampen, Phys. Rev. **91**, 1267 (1953).
[4] R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951).
[5] A. Ia. Povzner, Матем. сб. (Math. Collection) **32** (74), 1 (1953).
[6] A. Ia. Povzner, Dokl. Akad. Nauk SSSR **104**, 360 (1955).
[7] L. D. Faddeev, Вестник ЛГУ (Bulletin Leningrad State University) **7**, 126 (1956).
[8] L. D. Faddeev, Вестник ЛГУ (Bulletin Leningrad State University) **7**, 166 (1957).
[9] D. J. Wong and J. S. Toll, Annals of Phys. **1**, 91 (1957).
[10] M. G. Krein, Dokl. Akad. Nauk SSSR **105**, 433 (1955).
[11] V. A. Marchenko, Dokl. Akad. Nauk SSSR **104**, 695 (1955).
[12] I. Kay and H. E. Moses, Nuovo cimento **3**, 276 (1956).
[13] L. D. Faddeev, Dokl. Akad. Nauk SSSR (in press).
[14] H. E. Moses, Phys. Rev. **102**, 559 (1956).

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ROTATIONAL STATES OF NONAXIAL NUCLEI

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A theory is developed for the energy states of nuclei which have no axial symmetry, and for the electromagnetic transitions between these states. It is shown that the breakdown of axial symmetry, though it does not appreciably change the rotational states from those for axial nuclei, leads to the appearance of new energy states. Comparison of theory with experiment shows that the so-called γ-vibrational levels of even-even nuclei should be regarded as rotational levels. The levels of some nuclei with the spin sequence 0, 2, 2, 3 should be assigned to this same type.

INTRODUCTION

ON the basis of the uniform nuclear model of A. Bohr and Mottelson,[5,6] the present authors have investigated[1-4] the energy levels of nonspherical nuclei corresponding to collective excitations in which there is no breakdown of the axial symmetry of the nucleus. It was shown that the rotation-vibration energy of the collective excited states of the nucleus is a function of only two parameters — the frequency of the surface oscillations of the nucleus and the ratio of the equilibrium deformation to the amplitude of the zero-point vibrations. The question naturally arises as to the extent to which these results remain valid when one takes account of possible breakdown of axial symmetry of the nucleus.

The question of the breakdown of axial symmetry of the nucleus has already been discussed qualitatively in some papers.[5,7,8] Recently it has even become customary (cf. for example references 9 and 10) to assign certain excited nuclear