

## EXPERIMENTAL RESULTS

Our results are plotted in Table I and those of Gutowsky and Woessner<sup>5</sup> in Table II. Numerical comparison of theory with experiment for  $C_3F_5OH_2COOH$  gives

$$(R - 1)_{\text{theor}} = 0.147; (R - 1)_{\text{exptl}} = 0.29.$$

The theoretical calculation made here is very approximate. It is assumed that the value of the anisotropy of the chemical shift tensor of fluorine is of the order of the shift itself, i.e.,  $7 \times 10^{-4}$ . The correlation time was estimated from the Debye formula and taken equal to  $10^{-10}$  sec. The difference in results by a factor of two can be regarded as a good approximation of theory to experiment for the estimate given. From Tables I and II it is seen that the value of  $R$  decreases with increasing number of fluorine nuclei in a molecule in which the number of nuclei of another halide remains the same. This, apparently, is a consequence of the reduced screening of fluorine. An increase in the number of chlorine nuclei in a molecule leads to an increase in  $R$ . The relaxation mechanism in question, connected in all probability with the properties of chlorine atoms, is for the time being still not clear.

Thus, the theory of Gutowsky and Woessner can

explain only the not-too-large values of  $R$ . Very large values of  $R$  are observed, as a rule, in the presence of chlorine atoms in a molecule. Therefore, further development of the theory is required for the detailed explanation of the observed effects. The problem at hand is made difficult at the present time, however, by the fact that no complete theory of screening exists.

In conclusion, I express my thanks to F. I. Skripov and P. M. Borodin for their interest in the work and its evaluation.

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## ON THE DISTRIBUTION OF NUCLEON DENSITY IN NUCLEI

F. I. KLIGMAN

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The distribution of the orbital moments in nuclei is considered on the basis of the statistical nuclear model using various distribution functions of the nucleon density. It is shown that reasonable nucleon density distribution functions can be found, satisfying the experiments on scattering of fast electrons on nuclei, the saturation properties of nuclear forces, and the fundamental requirements of the shell model.

THE problem of nuclear structure, namely the distribution of nucleons in the nucleus, is being studied at present from different angles of approach.

Experimental and theoretical investigations of the scattering of fast electrons on nuclei<sup>1-4</sup> have shown that the proton density is approximately constant over the main part of the nuclear volume and

falls off smoothly at its boundary. The most characteristic feature of the distribution function, found as the result of the above investigations, is the fact that the thickness of the surface layer is constant — independent of the mass number  $A$  in a wide range of  $A$  — and that the mean (proton) radius of the nucleus varies as  $A^{1/3}$ ,<sup>4</sup> i.e. that there is no

affinity of proton distribution in various nuclei. Such a distribution, both for protons and neutrons, is consistent with the saturation properties of nuclear forces<sup>5-7</sup> and with the experiments on the scattering of fast nucleons on nuclei.<sup>8</sup>

The problem of the nucleon distribution is connected with the problem of the distribution of the orbital moments in the nucleus. A series of publications has been devoted to the latter subject<sup>7,9-14</sup> dealing, partly, with the determination of  $n_l$ , the number of nucleons with a given  $l$ . The results, obtained from statistical considerations, have been compared with those of the shell model. The assumed nucleon distribution is anisotropic, the density falling off according to a Gaussian<sup>7</sup> or exponential<sup>13,14</sup> law for  $r \geq R_0$  (where  $R_0$  is the region of constant density).

In the present work, the maximum number of nucleons of one kind (protons or neutrons), with orbital moment  $< l$ , present in a nucleus has been calculated by means of a statistical method, using various nucleon distribution functions. The different types of distributions are investigated for the correct interpretation of results on the scattering of fast electrons on nuclei, and the values  $N_l$  obtained are compared with these predicted by the shell model. Such a comparison is of a special interest, since the empirical values of  $N_l$  are given sufficiently accurately by the Mayer scheme, while, at the same time, a statistical (quasi-classical) treatment of the orbital moments can be applied for the calculation of  $N_l$  for heavy and even medium nuclei in view of the large values of  $l = l_{\text{lim}}$ . It is shown that the value of the functional  $K$ , which determines the value of  $N_l$  according to the statistical theory, and which is calculated on the basis of empirical values of  $N_l$  and  $l$  taken from the shell model, is approximately constant for all nuclei. At the same time, the value of  $K$  obtained by means of the statistical theory depends both on the nucleon distribution function and on the parameter of that function representing the ratio of the thickness of the surface layer to the radius  $R_0$  of the constant-density region (the "mean" nuclear radius). The value of  $K$  is constant when the above parameter remains constant, i.e., when there is an "affinity" of the nucleon den-

sity distribution in various nuclei. This, however, is not a necessary condition. The possibility is also discussed of selecting a nucleon distribution function leading to a value of  $K$ , calculated by the statistical method, constant for all nuclei, adjusting the parameter of the distribution function for agreement with the experimental results of electron scattering on nuclei.

It follows from the Fermi statistics<sup>15</sup> that the number of nucleons of one kind in a nucleus  $n_l$ , having the moment of momentum between  $l$  and  $l + 1$ , can be given by the expression

$$n_l = \frac{4}{h}(2l + 1) \int_{r_1}^{r_2} \left\{ \left( \frac{3h^3}{8\pi^2} \rho \right)^{2/3} - \frac{h^2}{4\pi^2} \frac{l(l+1)}{r^2} \right\}^{1/2} dr, \quad (1)$$

where  $\rho(r)$  is the density of nucleons of the given kind; the integration is extended over real values of the integrand. For the maximum value of  $l$  in the nucleus we have  $r_1 = r_2$  and  $n_l = 0$  which, according to Eq. (1), corresponds to the maximum of  $F(r) = r^3 \rho(r)$ .

The limiting value of  $l$  for particles of a given kind in a nucleus is, therefore, connected with the density distribution function  $\rho(r)$  of these particles, by the following relation (for sufficiently large  $l$ ):

$$[r^3 \rho(r)]_{\text{max}} = (2l + 1)^3 / 24\pi^2. \quad (2)$$

The function  $\rho(r)$  must satisfy also the condition

$$4\pi \int_0^\infty \rho(r) r^2 dr = N_l. \quad (3)$$

From Eqs. (2) and (3) we obtain

$$N_l = K(2l + 1)^3, \quad (4)$$

$$K = \frac{1}{6\pi} \int_0^\infty \rho(r) r^2 dr / [r^3 \rho(r)]_{\text{max}}. \quad (5)$$

Equations (4) and (5) are correct for sufficiently heavy spherical or almost spherical nuclei to which can be applied, on the one hand, the statistical method and, on the other, the notion of the orbital momentum of a particle characterized by an individual quantum number  $l$ . The second condition, however, is exactly the one necessary for the applicability of the shell model.

The values  $N$  and  $l$  based on the Mayer level scheme, and the corresponding empirical values of  $K_{\text{emp}} = N_l / (2l + 1)^3$ ,  $K$  being defined by Eqs. (4) and (5), are given in Table I. It can be seen that  $K$  is almost constant for all nuclei, decreasing slowly for heavier nuclei (larger  $N_l$ ).

For the calculation of  $K$  according to Eq. (5), let us consider the following distribution functions (similar for protons and neutrons):

TABLE I

$l$	$N_l$	$K_{\text{emp}}$
3	20	0.058
4	40	0.055
5	70	0.053
6	112	0.051

$$\rho(r) = \rho_0 = \text{const for } r \leq R_0; \quad (6)$$

$$\rho(r) = \rho_0 f\left(\frac{r-R_0}{a}\right) = \rho_0 f(x) \text{ for } r \geq R_0,$$

where  $f(x)$  is a decreasing function. We also consider functions of the Fermi type

$$\rho(r) = \rho_0 / [1 + e^{(r-c)/b}], \quad (7)$$

which are similar to (6) for  $b/c \ll 1$ .

We introduce the following definitions of the thickness of the surface layer  $d$  and of the mean nuclear radius  $R_C$ :<sup>4</sup>

$$d = r_{0.1} - r_{0.9}; \quad r = R_C \text{ for } \rho = \rho_0/2, \quad (8)$$

where  $r_{0.1}$  and  $r_{0.9}$  are the distances from the center of the nucleus, corresponding to the densities  $0.1\rho_0$  and  $0.9\rho_0$ ;  $\rho_0$  is the density in the central region of the nucleus.

According to Eq. (8),  $d$  is connected with the parameter  $b$  of Eq. (7) by the expression

$$d = b \ln 9 \approx 4.4 b. \quad (9)$$

The relation between  $d$  and the parameter  $a$  of Eq. (6) depends on the function  $f(x)$  in Eq. (6).

For the case of a Fermi-type distribution [Eq. (7)], we obtain from Eqs. (5) and (8) the following expression for  $K$

$$K = \frac{J}{6\pi x_m^3} (1 + e^{(x_m-1)/\varepsilon}), \quad \varepsilon = \frac{b}{c}, \quad (10)$$

$$J = \int_0^\infty \frac{x^2 dx}{1 + e^{(x-1)/\varepsilon}}, \quad x_m = \frac{r_m}{c},$$

where  $r_m$  is the value of  $r$  corresponding to the maximum of the function  $r^3\rho(r)$ .

From Eqs. (7) and (10) we have

$$x_m = \varepsilon y_m + 1, \quad (11)$$

where  $y_m$  is the root of the equation

$$e^{y_m} [y_m + (1/\varepsilon) - 3] = 3. \quad (12)$$

Eq. (12) was solved numerically for various values of  $\varepsilon$ , and the corresponding values of  $x_m$  were used to calculate  $K$  according to Eq. (10). The value of  $J$ , which enters the expression for  $K$ , was also determined for various  $\varepsilon$  by means of numerical integration.

Let us introduce a parameter  $\gamma$ , which describes the nucleon distribution. By definition:

$$\gamma = \frac{d}{R_{\text{eff}}}, \quad \frac{4}{3} \pi R_{\text{eff}}^3 \rho_0 = 4\pi \int_0^\infty r^2 \rho(r) dr. \quad (13)$$

For the case of the Fermi-type distribution (7) we have

$$R_{\text{eff}} = c(3J)^{1/3}, \quad \gamma = 4.4\varepsilon/(3J)^{1/3}. \quad (14)$$

TABLE II

$\varepsilon$	$\gamma$	$K$	$R_{\text{eff}}$	$A$
0.088	0.38	0.035	1.02c	197
0.120	0.51	0.038	1.04c	77
0.151	0.62	0.041	1.06c	40

Numerical integration of  $J$  yields a number of values of  $\gamma$  according to Eq. (14) and the relation between  $\gamma$  and  $K$ , the latter being defined by Eq. (10). The computed values of  $K$  for certain values of the parameter  $\varepsilon = b/c$  of the function (7), and the corresponding values of  $\gamma$ , are given in Table II. The corresponding values of  $A$  are also given in the table. The relation between the parameters  $\varepsilon$ ,  $\gamma$ , and  $A$  is given by Eq. (9) and (14), and the experimental values of  $d$  and  $c$  have been taken from reference 4, devoted to the scattering of fast electrons on protons:

$$d = (2.4 \pm 0.3) 10^{-13} \text{ cm}, \quad (15)$$

$$c = (1.07 \pm 0.02 A^{1/3}) 10^{-13} \text{ cm};$$

The relative error of  $\gamma$  and  $\varepsilon$  computed using the experimental values (15) is less than 15%. The values of  $R_{\text{eff}}$  calculated according to Eq. (14) and (15) are given in the fourth column of Table II. It can be seen that  $R_{\text{eff}}$  is not very different from  $c \approx R_C$ . The data in Table II indicate a low sensitivity of the value of  $K$ , calculated from the Fermi-type distribution function, with respect to the parameter  $\gamma$ .

The dependence of  $K$  on  $\gamma$  has also been calculated using the distribution function (6), for three types of density fall-off at  $r \geq R_0$ : (a) for a Gaussian fall-off:

$$\rho(r) = \rho_0 \exp\left\{-\left(\frac{r-R_0}{a}\right)^2\right\} \text{ for } r \geq R_0; \quad (16)$$

(b) for a linear fall-off

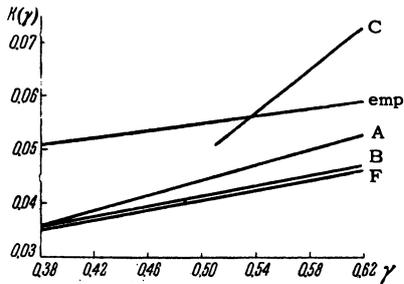
$$\rho(r) = \rho_0 (R_0 + a - r)/a \text{ for } R_0 \leq r \leq R_0 + a; \quad (17)$$

$$\rho(r) = 0 \text{ for } r \geq R_0 + a;$$

(c) for an exponential fall-off:

$$\rho(r) = \rho_0 \exp\{- (r - R_0)/a\} \text{ for } r \geq R_0. \quad (18)$$

The functions  $K(\gamma)$  obtained for the cases (a), (b), and (c) are represented graphically in the figure (curves A, B, and C, respectively). The function  $K(\gamma)$  based upon the Fermi-type distribution is also shown in the figure (curve F). For comparison of the results obtained with those of the shell model, the function  $K_{\text{emp}}(\gamma)$  based upon the relations  $K_{\text{emp}}(N_I)$  (Table I) and  $\gamma(A)$  (Table II) is included. For small  $N_I$  (20, 40, 70)



it has been assumed that  $A \approx 2Z$  (taking into account that  $N_Z$  is the number of either the protons or the neutrons, the value of  $A$  for real stable nuclei with  $N_Z = 70$  is in the range from 120 to 176); for  $N_Z = 112$  it has been assumed that  $A = 188$  (the mean of the values 186, 188, and 190 for stable nuclei with 112 neutrons). It should be noted that the spread of the values of  $A$  for  $N_Z = 70$  (120 to 176) corresponds to a variation of  $\gamma$  from 0.40 to 0.45.

It can be seen that in the case (a) and, especially, in the case (b),  $K$  is not sensitive to the variation of  $\gamma$ , as in the case of distribution (7): for  $\gamma$  changing from 0.38 to 0.62, the value of  $K$  varies between 0.036 and 0.048 in the case (a), between 0.036 and 0.042 in the case (c), and between 0.035 and 0.041 for the Fermi-type distribution. It has been shown, on the basis of Eqs. (6), (16), and (17) that, like in the case of the Fermi-type distribution,  $R_{\text{eff}}$  is not much different from  $R_c$  in the cases (a) and (b). Furthermore, it can be seen that the curves  $A$ ,  $B$ , and  $F$  are approximately parallel to  $K_{\text{emp}}(\gamma)$ , although the corresponding theoretical values of  $K$  are by  $\sim 30\%$  less than those of  $K_{\text{emp}}$ .

The result for the relation  $K(\gamma)$  in the case (c) is different. Curve  $C$  rises sharply, in contrast with the empirical curve, and the values  $\gamma < 0.51$ , which correspond to  $\alpha = a/R_0 < 1/3$  according to Eqs. (6), (8), (13), and (18) are not consistent with Eq. (5). At the same time, according to experimental results on electron scattering,<sup>4</sup> the values of  $\gamma > 0.51$  correspond to  $A < 77$ . The statistical determination of  $N_Z$  using the distribution function with an exponential density fall-off and with parameters determined from electron scattering phenomena is not applicable to heavy nuclei for which, in general, the statistical method is most appropriate. It should be noted also that, independently of the value of the parameters determined from electron scattering experiments, the condition  $\alpha \geq 1/3$  for the function with an exponential density fall-off requires, according to Eq. (8), that the minimum value of the ratio of

the surface layer thickness  $d$  to the radius of the constant density region  $R_0$  were greater than 0.73. For other cases, the above ratio is either unrestricted [case (a)], or sufficiently small [0.23 for the case (b)].

The Fermi-type distribution function and the distribution functions with a Gaussian or linear fall-off at  $r \geq R_0$ , which describe correctly the scattering of electrons on nuclei,<sup>2,4</sup> lead to a value of  $K$  that is almost constant and which varies with  $\gamma$  in an analogy to the empirical dependence  $K(\gamma)$  corresponding to the shell model, although the theoretical curves lie somewhat lower than the empirical. It can be seen also that the distribution function with an exponential fall-off leads to a discrepancy with the experiment.

In conclusion we would like to mention that, although neither theoretical nor experimental investigations give at present the exact shape of the distribution function of the nucleons in nuclei, the present work indicates a possibility of application of the statistical method of determining  $N_Z$  to the selection of a distribution function. The  $N_Z$  found satisfy both the requirements of the shell model ( $K$  independent of  $A$ ) and the saturation property of nuclear forces, as well as the experiments on scattering of electrons by nuclei (i.e., the requirement that the effective nuclear radius be proportional to  $A^{1/3}$  while the thickness of the surface layer be independent of  $A$ ).

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## ON THE ELECTRON CAPTURE MECHANISM AND THE CURRENT LIMIT IN BETATRONS

A. N. MATVEEV

Moscow State University

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Mechanism of electron capture in betatrons is discussed. Its basis is the Coulomb interaction of the electrons in the beam and the losses of electrons to the walls of the doughnut. The problem is treated exactly for a simplified model. It is shown that the considered capture mechanism has a high effectiveness which is in agreement with experiment. An expression for the limiting current, valid also for relativistic energies, is given.

### 1. INTRODUCTION

NO existing theory explains the capture of electrons in betatrons, nor has a satisfactory physical picture of this process been developed. However, several important experiments<sup>1-6</sup> performed during the last years have clarified this problem to a considerable extent. They have confirmed that the capture of electrons in betatrons is due to their collective interactions. It is therefore unnecessary to consider one-electron theories of electron capture.<sup>7-8</sup> Their applicability is limited to rather small injection currents.

One can subdivide the capture mechanisms based on collective interactions into three groups: (a) mechanisms connected with the action of self-induction of the non-stationary electron current in the doughnut;<sup>9-10</sup> (b) mechanisms based on the interaction of the electrons with the Coulomb field of the space charge;<sup>10, 11, 13-15</sup> (c) statistical capture mechanisms.<sup>16</sup>

Experimental models have shown that the mechanism based on the self induction of the non-stationary current<sup>4</sup> cannot explain, at the actual strengths of the injection currents, the observed effectiveness of the capture and does not play an important part in the capture process. Its effectiveness is similar to that of the adiabatic contraction of the orbit and of the adiabatic damping of the betatron oscillations. Thus, one

can consider it to be sufficiently well established at present that the induction-type mechanism does not play an essential part in the overall picture of the electron capture.

As to the statistical capture mechanism, it has been shown earlier<sup>17</sup> that it can work only at sufficiently small injection currents, in the region between the single-electron capture and the collective capture. At such injection currents where the capture process is particularly effective, this mechanism does not play an essential part.

It appears thus that the most likely injection mechanism is that which takes into account the Coulomb interaction of the electrons. The different effects associated with the Coulomb interaction at injection time are discussed in references 10, 11, and 13 to 15. The mechanism treated by Wideroe is based on the energy lost by electrons passing through a space-charge cloud whose charge density decreases in time. This process cannot be decisive since it does not explain the capture on the leading side of the injection pulse. Bardeen has proposed a mechanism based on the use of the azimuthal inhomogeneity of the space charge. This mechanism is in disagreement with the well-known fact that the capture works equally well with injection from the outside (where  $n \approx 1$ ) as from the inside (where  $n \approx 0.5$ ). It also is in disagreement with the fact that if