ON THE THEORY OF THE INTERACTION OF FAST NEUTRONS OF VARIOUS ANGULAR MOMENTA WITH SEMI-TRANSPARENT NUCLEI

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The participation of fast neutrons with various angular momenta in the interaction with semi-transparent nuclei is studied within the framework of the complex square-well model for the nucleus. The exact expressions for the partial cross sections are approximated with the help of special asymptotic formulas for the cylindrical functions which are valid throughout the momentum range of interest. This approximation allows a more precise determination of the participation of waves with angular momentum $l \sim kR$ in neutron scattering and absorption processes as compared with the results obtained in the quasi-classical approximation. Corrections to the integral absorption and scattering cross sections are obtained, which are important for high and low effective absorption.

In many recent papers[1-4] on the scattering of particles from nuclei use is made of the model of a complex potential well. With this model it was possible to determine accurately the dependence of the cross section on energy in both the low- and high-energy regions. For small neutron energies only the wave with low angular momenta participate in the interactions with the nucleus. In the calculation of the cross section one can therefore sum directly over the partial cross sections.

For high neutron energies many waves (with momenta $l \lesssim kR$), interact with the nucleus, and for the calculation of the cross section it is necessary to sum over the whole contributing interval of momenta, and to determine accurately the upper limit of this interval. In addition, the diffuseness of the nuclear boundary also affects the value of the cross section.

Fernbach, Serber, and Taylor[1] regard the nucleus as a homogeneous sphere with a certain absorption coefficient and a certain refraction index. Applying a quasi-classical method based on an optical analogy (in the following we shall call this the optical approximation) to this model, it was possible to fit satisfactorily the neutron scattering experiments for energies of order 100 Mev. However, the differential scattering cross sections came out incorrectly. As shown by Pasternack and Snyder,[5] this discrepancy is for the most part due to the inaccuracy of the optical approximation, and not due to the model. The optical approximation gives the true partial cross sections only for waves with momenta $l \lesssim kR/2$. In the region of momenta $l \lesssim kR$ the values for the partial cross sections appear to be too low, and waves with momenta $l > kR - 1/2$ are not accounted for at all.

Drozdov[3] obtained an expression for the absorption cross section for fast neutrons, solving the Schrödinger equation approximately with the help of the semi-classical method of Petrashen'.[6] This paper improves on the results of the crude optical approximation, since the Petrashen' method gives good accuracy in the region $l < kR$; however, Drozdov applies without justification the formulas thus obtained also in the region $l \sim kR$ and cuts off the integration at the momentum $l = kR - 1/2$, arguing that only waves with momentum corresponding to an impact parameter smaller than the nuclear radius participate in scattering and absorption processes.

Thus both methods[1,3] give a good approximation only for partial waves with momenta $l < kR$. The momentum region $l \sim kR$ is treated incorrectly, and in the calculation of the cross section the integration is cut off at the momentum $l = kR - 1/2$. It is therefore of interest to investigate in more detail the role of the momenta $l \sim kR$ in the interaction of neutrons with the nucleus, and to determine the behavior of the scattering and absorption coefficients in this momentum region. This paper treats this question on the basis of the simpler model of a complex square potential well. This well, for which the Schrödinger equation can be solved exactly, permits an estimate of the accuracy of the results obtained. In addition, the effect of the diffuseness of the nuclear boundary is excluded.

1. REGION OF SMALL MOMENTA

With the model of a square complex potential well it is appropriate not to start from an approximate solution of the Schrödinger equation, but to obtain the exact expressions for the scattering and absorption coefficients. In summing these we use more exact approximations, taking account of all waves participating in the interaction.

If the interaction potential of the neutron-nucleus system has the form $U(r) = -V - iW$ for $r \leq R$ and $U(r) = 0$ for $r > R$, where R is the nuclear radius, the Schrödinger equation for the radial neutron wave function with momentum l and energy E can be solved exactly. The reflection coefficient has the form:

$$\beta_l = -\frac{xJ_{l+1/2}(z)\,H^{(2)}_{l-1/2}(x) - zJ_{l-1/2}(z)\,H^{(2)}_{l+1/2}(x)}{xJ_{l+1/2}(z)\,H^{(1)}_{l-1/2}(x) - zJ_{l-1/2}(z)\,H^{(1)}_{l+1/2}(x)}, \qquad (1)$$

$$x = kR = \sqrt{2m}\,E^{1/2}R/\hbar, \qquad z = \sqrt{2m}\,(E + V + iW)^{1/2}R/\hbar,$$

where m is the reduced mass of the neutron-nucleus system.

In order to get from (1) expressions for the coefficients of scattering, $|1-\beta_l|^2$, and absorption, $1-|\beta_l|^2$, which are convenient for inspection and for the integration over the whole contributing momentum interval, we have to use approximate formulas for the cylindrical functions. However, the usual formulas for $l \ll x$ and $l \sim x$ are applicable only in a rather narrow region of momenta: to those of order $x^{1/2}$ for small momenta, and to those below $x^{1/3}$ for $l \sim x$. Below we shall therefore use special formulas for the cylindrical functions in the region $l < x$ (see reference 7), and the formulas of Fock[8] for the region $l \sim x$.

In the region $l < x$ the asymptotic formulas used, e.g. for the Bessel functions, have the form:

$$J_\nu(\nu\sec\beta) = \sqrt{\frac{2}{\pi\nu\tan\beta}}\,[\cos(\nu\tan\beta - \nu\beta) + O(1/\nu\tan\beta)].$$

The retention of only the first term in this expansion gives an error of order $1/3\nu\ \tan\beta$; in the momentum region $l < 3x/4$ this error does not surpass 5%. This approximation gives for the reflection coefficient the formula

$$\beta_l = \exp\Big\{2i\Big[\sqrt{z^2-(l+1/2)^2} - \sqrt{x^2-(l+1/2)^2} + (l+1/2)\Big(\cos^{-1}\frac{l+1/2}{x} - \cos^{-1}\frac{l+1/2}{z}\Big)\Big]\Big\}, \qquad (2)$$

which agrees with the result obtained by Drozdov.[3]

For high energies, where $\eta \equiv \operatorname{Im} z \ll x$ and $\xi \equiv \operatorname{Re} z - x \ll x$, the phase at infinity δ_l, which is related to the reflection coefficient by $\beta_l = e^{2i\delta_l}$, has the form:

$$\delta_l = (\xi + i\eta)\Big[1 - \frac{1}{2}\Big(\frac{l+1/2}{x}\Big)^2 - \frac{1}{8}\Big(\frac{l+1/2}{x}\Big)^4 - \cdots\Big] \approx (\xi + i\eta)\sqrt{1-(l+1/2)^2/x^2}.$$

In the optical approximation[1]

$$\delta_l = [(n-1)x + iKR/2]\sqrt{1-(l+1/2)^2/x^2}, \qquad (3)$$

where K is the absorption coefficient, and n the refraction index. At high energies we have thus the following correspondence between the parameters: η corresponds to $KR/2$, and ξ corresponds to $(n-1)x$. This enables us to compare the results of the optical approximation with ours. Since η depends not only on the imaginary part W of the potential, but also on the real part V, on the neutron energy, and on the mass of the nucleus, we shall in the following call this parameter the "effective absorption." Analogously, we shall call ξ the "effective refraction."

2. REGION OF INTERMEDIATE MOMENTA

Of special interest is the momentum region $l \sim x$. For an approximation of expression (1) in this momentum region we use the asymptotic formulas for the cylindrical functions of Fock.[8] For example, for the Bessel function we have:

$$J_{l+1/2}(z) = \frac{1}{\sqrt{\pi}}\Big(\frac{z}{2}\Big)^{-1/3} \times \Big\{v(\tau) - \frac{1}{60}\Big(\frac{z}{2}\Big)^{-2/3}[\tau^2 v'(\tau) + 4\tau v(\tau)] + \cdots\Big\}. \qquad (4)$$

With the help of these formulas the reflection coefficient takes the form

$$\beta_l = \frac{xv(\tau)\,w^*(t') - zv(\tau')\,w^*(t)}{xv(\tau)\,w(t') - zv(\tau')\,w(t)}, \qquad (5)$$

where

$$w(t) = \sqrt{\pi/3}\,e^{2\pi i/3}(-t)^{1/2}H^{(1)}_{1/3}[2/3(-t)^{3/2}] = u(t) + iv(t),$$

$$w^*(t) = u(t) - iv(t),$$

$$t = (l+1/2-x)/(x/2)^{1/3}, \quad t' = (l-1/2-x)/(x/2)^{1/3},$$

$$\tau = (l+1/2-z)/(z/2)^{1/3}, \quad \tau' = (l-1/2-z)/(z/2)^{1/3}.$$

In formula (5) we retain only the first terms of expansion (4). This approximation introduces an error of order $|t|^{5/2}/60\,(x/2)^{2/3}$; in the region of momenta $|l-x| \lesssim x^{1/2}$ this error does not surpass 2%. Hence formula (2) gives a good approximation for the reflection coefficient in the region $l < x$, while formula (5) applies to the region $l \sim x$. Thus the whole contributing momentum interval is covered.

We calculate the scattering and absorption coefficients with the help of formula (5) and expand the resulting expression in powers of $(x/2)^{1/2}$ We obtain:

$$|1-\beta_l|^2 = 4(\xi^2+\eta^2)\frac{[v'(t)^2 - tv(t)^2]^2}{(x/2)^{2/3}}, \tag{6}$$

$$1-|\beta_l|^2 = 4\eta\frac{v'(t)^2 - tv(t)^2}{(x/2)^{1/3}}. \tag{7}$$

In going from negative to positive values of the argument, the Airy function $v(t)$ changes from an oscillating function to an exponentially decreasing function. Therefore the scattering and absorption coefficients decrease rapidly after some characteristic value $l_0 \sim x - 1/2$. However, l_0 does not in general equal $x - 1/2$: for absorption coefficients with high effective absorption ($\eta > (x/2)^{1/3}$) $l_0 \gtrless x - 1/2$, for those with low effective absorption ($\eta \ll 1$) $l_0 < x - 1/2$. It was found that the waves with momentum $l > x - 1/2$ contribute significantly to the absorption cross section for high effective absorption (see Fig. 1): their inclusion gives a correction of order 20% at relatively low energies ($x \sim 10$, $\eta \sim 2$). For small effective absorption ($\eta \ll 1$) the partial cross sections begin to decrease rapidly already for $l < x - 1/2$ (see Fig. 2), so that the approximations of references 2 and 3 give these incorrectly in the region $l \lesssim x$. As a result, these approximations yield too high values for the cross section (of order 20% for $x \sim 10$). It was found that the region $l > x - 1/2$ contributes nothing to the partial cross sections for all conceivable values of the effective absorption and refraction. A comparison is possible only with the optical approximation, since Drozdov[3] investigated only the absorption of neutrons. We shall show below that our results differ from those of the optical approximation only for small effective absorption and refraction. The optical approximation gives somewhat larger values in this case.

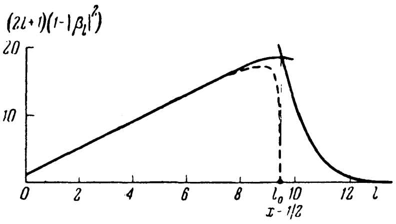

FIG. 1. Partial absorption cross sections for the parameters $x = 10$, $\xi = 1$, $\eta = 1.875$; the solid curves are obtained with formulas (2) and (7), the dotted curve is obtained with the optical approximation (3).

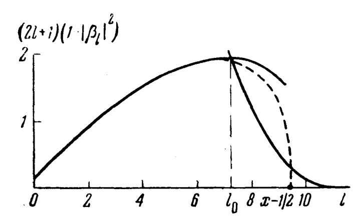

FIG. 2. Partial absorption cross sections for the parameters $x = 10$, $\xi = 1$, $\eta = 0.05$; the solid curves are obtained with formulas (2) and (7), the dotted curve is obtained with the optical approximation (3).

3. CROSS SECTIONS FOR SCATTERING AND ABSORPTION

With the approximations obtained for the coefficients of reflection, scattering, and absorption it is possible to calculate the cross sections by integrating* over the whole momentum interval (separately for $l \le l_0$ and for $l \ge l_0$). Integrating, we obtain:

$$\frac{\sigma_s}{\pi R^2} = \int_0^\infty (2l+1)|1-\beta_l|^2 dl = \left[1 + \frac{t_0}{(x/2)^{2/3}} + \cdots\right] \times \left\{1 + e^{-4\eta}[1+\eta+\cdots] - 2e^{-2\eta}\left[\cos 2\xi\left(1+\frac{\eta}{2}+\cdots\right) + \sin 2\xi\left(\frac{\xi}{2}+\cdots\right)\right]\right\} + \frac{4(\xi^2+\eta^2)}{(x \quad ^{4/3}}(0.011 - 0.044t_0 + \cdots), \tag{8}$$

$$\frac{\sigma_c}{\pi R^2} = \int_0^\infty (2l+1)(1-|\beta_l|^2)\,dl = \left[1 + \frac{t_0}{(x/2)^{2/3}} + \cdots\right] \times \left\{1 - e^{-4\eta}[1+\eta+\cdots] + \frac{8\eta}{x}[0.096 - 0.21\ t_0 + \cdots]\right\}, \tag{9}$$

where t_0 is related to the characteristic l_0 by

$$t_0 = (l_0 + 1/2 - x)/(x/2)^{1/3},$$

and is separately determined, for the scattering coefficient from the identification of the expression $|1 - \beta_l|^2$ in formulas (2) and (6), and for the absorption coefficient from the identification of the expression $1 - |\beta_l|^2$ in formulas (2) and (7). For example, for $\eta \ll 1$ and $\xi \ll 1$, t_0 is the same for the scattering and for the absorption, and has the form

$$t_0 = 0.685 - 1.315\,(x/2)^{1/3} + 0.1/(x/2)^{1/3} - \cdots, \tag{10}$$

while, for large η, t_0 for the absorption is

$$t_0 = 0.53 - 0.63\,(x/2)^{1/3}/\eta + \cdots. \tag{11}$$

For comparison with the results of the optical approximation and with the approximation of Drozdov, we consider formulas (8) and (9) for,

*The relative error introduced by changing the summation over the partial scattering and absorption cross sections (to be performed in the calculation of the cross section) into an integration is very small ($\sim x^{-2}$) and can be neglected.

the cross sections, expressed in terms of the effective absorption η and the effective refraction ξ, in the limiting cases. Depending on the relation between the neutron energy and the parameters of the potential well the expressions for η and ξ have the following form:

for $W < V < E$ $(x > y,\ \zeta < 1)$:

$$\eta = \sqrt{\frac{mR^2}{2\hbar^2 E}}\, W\left(1 - \frac{V}{2E} + \cdots\right) = \frac{\zeta y^2}{2x}\left(1 - \frac{y^2}{2x^2} + \cdots\right),$$
$$\xi = \sqrt{\frac{mR^2}{2\hbar^2 E}}\, V\left(1 - \frac{V}{4E} + \cdots\right) \tag{12}$$
$$= \frac{y^2}{2x}\left(1 - \frac{(1-\zeta^2)\, y^2}{4x^2} + \cdots\right);$$

for $W < E < V$ $(\zeta y^2 < x^2 < y^2)$:

$$\eta = \sqrt{\frac{mR^2 V}{2\hbar^2}}\, W\left(1 - \frac{E}{2V} + \cdots\right) = \frac{\zeta y}{2}\left(1 - \frac{x^2}{2y^2} + \cdots\right),$$
$$\xi = \sqrt{\frac{2mR^2 V}{\hbar^2}}\left(1 - \sqrt{\frac{E}{V}} + \frac{E}{2V} - \cdots\right) \tag{13}$$
$$= y\left(1 - \frac{x}{y} + \frac{x^2}{2y^2} - \cdots\right),$$

where

$$y = \sqrt{2m}\, V^{1/2} R/\hbar, \quad \zeta = W/V.$$

For small effective absorption $(\eta \ll 1)$ we obtain from (9), (10), and (12):*

$$\frac{\sigma_c}{\pi R^2} = 2.72\eta\left(1 - \frac{1.86}{x^{1/2}} + \frac{1.48}{x^{2/3}} - \cdots\right)$$
$$= 1.36\,\frac{\zeta y^2}{x}\left(1 - \frac{1.86}{x^{1/2}} + \frac{1.48}{x^{2/3}} - \cdots\right); \tag{14}$$

in the same case references 1 and 3 give, respectively:

$$\sigma_c/\pi R^2 = 2.67\eta, \quad \sigma_c/\pi R^2 = 2.72\eta.$$

We see that the optical approximation and Drozdov's results give too large values for the cross section (with a relative error of order $-0.6x^{-1/2}$).

For large effective absorption $(\eta > (x/2)^{1/3})$ we obtain from (9), (11), and (13):

$$\frac{\sigma_c}{\pi R^2} = 1 + 1.66\frac{\eta}{x} - 0.63\frac{1}{\eta (x/2)^{2/3}} + \cdots$$
$$= 1 + 0.83\frac{\zeta y}{x} - 0.415\frac{\zeta x}{y} - \frac{1.59}{\zeta y x^{2/3}} + \cdots; \tag{15}$$

in references 1 and 2 $\sigma_c/\pi R^2 = 1$. The correction to the absorption cross section is in this case $\sim \eta/x$.

As mentioned above, for the scattering cross section we are interested in the results for small effective absorption and refraction $(\eta \ll 1,\ \xi \ll 1)$; we find from (8), (10), and (12) (see footnote*):

$$\frac{\sigma_s}{\pi R^2} = 2(\xi^2 + \eta^2)\left(1 - \frac{1.86}{x^{1/2}} + \frac{0.865}{x^{2/3}} - \cdots\right)$$
$$= \frac{y^4}{2x^2}(1 + \zeta^2)\left(1 - \frac{1.86}{x^{1/2}} + \frac{0.865}{x^{2/3}} - \cdots\right), \tag{16}$$

while reference 1 gives

$$\sigma_s/\pi R^2 = 2(\xi^2 + \eta^2).$$

For a graphical illustration of the accuracy of the above calculations see Fig. 3. The points represent the exact values of the absorption coefficient for the parameters $\xi = 1$, $\eta = 0.5$, $x = 10$;

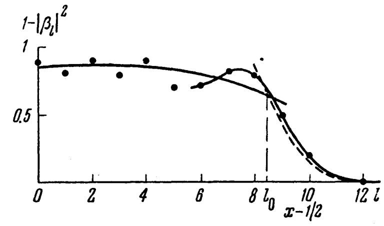

FIG. 3. Absorption coefficients for the parameters $x = 10$, $\xi = 1$, $\eta = 0.5$; the points denote the exact values, the solid curves are obtained with the formulas (2) and (5), the dotted curve is obtained with the approximate formula (7).

the solid curve represents the values of the absorption coefficient calculated with the approximate formulas (2) in the region $l < x$, and (5) in the region $l \sim x$. In addition, in the region $l \sim x$ the dotted curve represents the absorption coefficient calculated with formula (7), i.e., retaining only the first term in expansion (5). We see from Fig. 3 that the approximate values are close to the exact ones. Although in region $l < x$ the approximation gives only a smooth curve for the absorption coefficient, the error in the calculation of the cross section is found to be insignificant. The absorption coefficients are integrated over the momenta (with weights $2l + 1$). Estimates show that the accuracy of the cross section calculations is on the order of 1 or 2%; thus the corrections to the cross sections obtained with the optical approximation and by Drozdov are greater than the calculational error by an order of magnitude. We note that our asymptotic expressions tally well (in the mean) in the region $l \lesssim l_0$ (see Fig. 3; in Figs. 1 and 2 the region of overlapping of the asymptotic curves is not shown).

CONCLUSION

Let us estimate the neutron energies for which the above results are valid. We base our estimate of the upper limit on the assumption that the neu-

*We emphasize in relation to formulas (14) to (16) that the inclusion of the region of intermediate momenta changes somewhat the dependence of the cross sections on energy.

tron-nucleus interaction can be summarily described with the model of a complex potential only for energies corresponding to a wavelength of the same order or larger than the average distance between the nucleons in the nucleus. Then this limit is in the region of energies of order 100 Mev. In addition, the asymptotic formulas for the cylindrical functions used in this paper are valid only for sufficiently large arguments $(x \ll 1)$. This limits the applicability of the above results at the low energy side. Our results refer thus to energies of 20 to 200 Mev.

We note further that for an accurate calculation of the cross sections in the framework of the complex potential well model one must, together with the effect of the intermediate momenta, also consider the effect of the diffuseness of the nuclear boundary. Both effects are included in the paper of Nemirovskii.[4] His numerical treatment of the formulas containing the cylindrical functions allows him to circumvent the difficulties connected with the approximation of these functions in the region of intermediate momenta. However, the great number of approximations in his paper makes it difficult to estimate the accuracy of his results.

The comparison of the results of the present paper with the results of Nemirovskii shows that the consideration of the diffuseness of the nuclear boundary always leads to an increase in the absorption cross section, while the inclusion of the intermediate momenta leads to an increase in the cross section for large effective absorption, but to a decrease for small effective absorption. Hence, for neutron energies of order 100 Mev and small ζ the two effects work in opposite directions. The increase of the cross section due to the diffuseness of the nuclear boundary (for diffuseness $\gtrsim k^{-1}$)* is of order 10% (for heavy nuclei).[4] On the other hand, we have shown above that the decrease in the cross section due to the proper inclusion of the intermediate momenta for a nucleus with sharp boundary is of the same order of magnitude. As a result, the proper inclusion of the region of intermediate momenta for neutron energies of order 100 Mev is just as essential as the consideration of the effects of the diffuseness of the nuclear boundary.

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*The comparison of the effects of the region of intermediate momenta and of the diffuseness of the nuclear boundary is made difficult by the absence of experimental data on the diffuseness parameter for $E \gtrsim V$.

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