MOMENT OF INERTIA OF NONSPHERICAL NUCLEI

Iu. K. KHOKHLOV

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor February 24, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 240-243 (July, 1958)

The formula proposed by Inglis for the nuclear moment of inertia is calculated by perturbation theory in an approximation that is quadratic in the deformation parameter. An infinite rectangle well is assumed. The obtained moment of inertia of closed shells is several times larger than the hydrodynamic moment of inertia.

1. Assume a nonspherical potential well, rigidly coupled to a massive rotator. If we fill the lower levels of the well with non-interacting nucleons, we obtain a certain system (called the Inglis model[1]), the rotary properties of which are of procedural interest in the study of the rotation of real nuclei. The moment of inertia of the system comprises the moment of inertia of the rotator and the moment of the inertia of the nucleons entrained by the rotator. The latter moment is given by formula (4), previously investigated by Inglis[1,2] and by Bohr and Mottelson.[3] These authors used the potential of the anisotropic harmonic oscillator, and were thus able to calculate (4) in closed form.. In the case of closed shells, the result obtained does not differ substantially from the hydrodynamic result.

The present paper concerns results obtained for an infinite rectangular potential well. Because of the great mathematical difficulties, we restrict ourselves to closed shells. The wave functions were calculated by perturbation of the boundary conditions, up to and including the fourth approximation. The perturbation due to the rotation was considered [see Eq. (4)] in the first order of perturbation theory. Alltogether the moment of inertia of the closed shells was determined up to terms quadratic in the deformation parameter.

2. In a coordinate system that rotates, together with the nucleus, around the x axis with angular velocity Ω, the boundary S of the potential well is given by

$$r = R_0(1+\alpha), \text{ where } \alpha = \sum_{\lambda=0} \alpha_\lambda P_\lambda(\cos\theta). \tag{1}$$

Here α_λ are coefficients which will be considered in the following small compared to unity, θ is the polar angle, and $P_\lambda(\cos\theta)$ are Legendre polynomials with the usual normalization $P_\lambda(1) = 1$.

The wave function Ψ obeys inside S the time-independent Schrödinger equation of a free particle, augmented by a term describing the Coriolis force

$$\left(\nabla^2 + k^2 + \frac{2}{\hbar} M\Omega L_x\right)\Psi = 0, \tag{2}$$

with the boundary condition

$$\Psi|_S = 0. \tag{3}$$

Here k is the wave number, M the nucleon mass, and L_x the projection of the angular momentum on the x axis.

The contribution to the moment of inertia from the given nucleon equals

$$I = \hbar\left[\frac{\partial}{\partial\Omega}(\Psi, L_x\Psi)\right]_{\Omega=0} = \frac{2M}{k^2}\frac{(\varphi, L_x\varphi)}{(\psi, \psi)}. \tag{4}$$

Here

$$\psi = \sqrt{N}\Psi_{\Omega=0}; \quad \varphi = \sqrt{N}\frac{\hbar k^2}{M}\frac{\partial\Psi}{\partial\Omega}\bigg|_{\Omega=0}; \quad N = (\psi, \psi). \tag{5}$$

The functions ψ and φ obey the following equations and boundary conditions

$$(\nabla^2 + k^2)\psi = 0; \quad (\nabla^2 + k^2)\varphi = -2k^2 L_x\psi; \tag{6}$$

$$\psi|_s = 0; \quad \varphi_s = 0. \tag{7}$$

In an unperturbed spherically-symmetrical square well the functions ψ have the form

$$\psi^{(0)} = j_l(k_0 r) Y_{lm}. \tag{8}$$

Here Y_{lm} are spherical harmonics, $j_l(x)$ the spherical Bessel functions, and k_0 the eigenvalue determined by the condition $j_l(k_0R_0) = 0$. In the following we shall denote $k_0R_0 \equiv X$. We shall characterize the excited states by the same numbers, l and X. We shall speak about closed l-shells in the nonspherical well in the sense that the participating states form closed l-shells for vanishing deformation.

In general ψ and φ are given by

$$\psi = \sum_{l'} a_{l'} j_{l'}(x) Y_{l'm}, \quad a_l = 1; \tag{9}$$

$$\varphi = \frac{1}{2}(\varphi^{(+)} + \varphi^{(-)}), \tag{10}$$

$$\varphi^{(+)} = \sum_{l'} [a_{l'} f_{l'}(x) + b_{l'} j_{l'}(x)] B_{l'} Y_{l', m+1},$$

$$\varphi^{(-)} = \sum_{l'} [a_{l'} f_{l'}(x) + b_{l'} j_{l'}(x)] C_{l'} Y_{l', m-1}. \tag{11}$$

Here

$$B_l = (L_+)_{m+1, m} = \sqrt{(l+m+1)(l-m)};$$

$$C_l = (L_-)_{m-1, m} = \sqrt{(l+m)(l-m+1)};$$

$$L_\pm = L_x \pm iL_y; \quad x = kr;$$

the numbers k, $a_{l'}$, $b_{l'}$, and c_l obviously depend on l, m, and X.

It is easy to see that the terms which contain the functions $f_{l'}(x) \equiv x \frac{d}{dx} j_{l'}(x)$ are particular solutions of the inhomogeneous equation for φ, while the terms containing $j_{l'}(x)$ are solutions of the corresponding homogeneous equation. The coefficients $b_{l'}$ are determined by the boundary condition $\varphi^{(+)}|_S = 0$ and the coefficients $c_{l'}$ are obtained from the $b_{l'}$ by changing m to −m.

The moment of inertia is now given by

$$I = \frac{1}{2}(I_+ + I_-); \quad I_+ = \frac{M}{k^2} \frac{(\psi, L_-\varphi^{(+)})}{(\psi\psi)} \tag{12}$$

$$= \frac{M}{k^2} \frac{\left(\sum_{l'} a_{l'} j_{l'}(x) Y_{l'm}, \ \sum_{l'} [a_{l'} f_{l'}(x) + b_{l'} j_{l'}(x)] B_{l'}^2 Y_{l'm}\right)}{\left(\sum_{l'} a_{l'} j_{l'}(x) Y_{l'm}, \ \sum_{l'} a_{l'} j_{l'}(x) Y_{l'm}\right)}.$$

The moment I_- follows from I_+ by the substitution of m by −m. We therefore do not have to investigate I_- separately, and correspondingly do not need the function $\varphi^{(-)}$.

We now expand all quantities in power series of α_λ:

$$k \approx k_0 + k_1 + k_2 + k_3 + k_4;$$

$$a_{l'} \approx a_{l'}^{(1)} + a_{l'}^{(2)} + a_{l'}^{(3)} \text{ for } l' \neq l; \quad a_l = 1; \tag{13}$$

$$b_{l'} \approx b_{l'}^{(-1)} + b_{l'}^{(0)} + b_{l'}^{(1)} + b_{l'}^{(2)}.$$

We have here written the terms needed to calculate the moment of inertia in the quadratic approximation. All these terms are successively determined by the boundary conditions (7).

The moment I_+ can be written as the series

$$I_+ \approx I_+^{(-1)} + I_+^{(0)} + I_+^{(1)} + I_+^{(2)}. \tag{14}$$

In summing over m, the terms $I_+^{(-1)}$, $I_+^{(0)}$ and $I_+^{(1)}$ vanish. Thus, as could be expected, the expansion of the moment of inertia for closed shells begins with the quadratic term:

$$I_{l,X}^{(2)} = \sum_m I_+^{(2)}. \tag{15}$$

After extensive calculations we obtain

$$I_{l,X}^{(2)} = \sum_m \frac{M}{k_0^2} \Big\{ \frac{1}{12} [(2l-1)(2l+5) + 2X^2] B_l^2 (p_{ll} - q_{ll})^2 + \sum_{l'}{}' \Big[\frac{1}{4}(B_l^2 + B_{l'}^2)(p_{ll'}^2 + q_{ll'}^2) - B_l B_{l'} p_{ll'} q_{ll'} \Big] \times [2X^2 + (1 + 2\varphi_{l'})\psi_{l'}] \Big\}. \tag{16}$$

Here

$$\varphi_{l'} = 1 + f_{l'}(X) / j_{l'}(X), \quad \psi_{l'} = l'(l'+1) - X^2 + \varphi_{l'} - \varphi_{l'}^2;$$

$$p_{ll'} = (l, m|\alpha|l', m); \quad q_{ll'} = (l, m+1|\alpha|l', m+1). \tag{17}$$

Taking for α the particular value

$$\alpha = \alpha_2 P_2(\cos\theta) \tag{18}$$

and summing over m, we have

$$I_{l,X}^{(2)} = \frac{9}{10} M R_0^2 \alpha_2^2 \times \Big\{ \frac{2}{9} \Big[1 + \frac{(2l-1)(2l+5)}{2X^2}\Big] \frac{l(l+1)(2l+1)(2l+5)}{(2l-1)(2l+3)^2} + 2X^4 \Big[\frac{l(l-1)}{(2l-1)^4} - \frac{(l+1)(l+2)}{(2l+3)^4}\Big] + X^2 \Big[\frac{l(l-1)}{(2l-1)^2}\Big(\frac{4}{2l-1} - 1\Big) + \frac{(l+1)(l+2)}{(2l+3)^2}\Big(\frac{4}{2l+3} + 1\Big)\Big]\Big\}. \tag{19}$$

The moment of inertia of the whole nucleus equals

$$I \approx g \sum_{l,X} I_{l,x}^2. \tag{20}$$

Here g = 4 is the number of nucleons which can simultaneously occupy the state with given l, X, and m.

3. Table I lists the moments of inertia of several nuclei obtained by numerically summing (19), in terms of the hydrodynamic moment of inertia I_{hydr} which is given in the quadratic approximation by

$$I_{\text{hydr}} = \frac{9}{10} M A R_0^2 \alpha_2^2; \tag{21}$$

here A is the atomic number of the nucleus.

We see that in the considered range of A the ratio I/I_{hydr} is several times larger than unity. Thus the results obtained in references 1 to 3 do not occur in our case.

Concerning the region of applicability of the quadratic approximation, we can make the following remarks. As was known earlier and confirmed in the present paper, the range of applicability of perturbation theory is rather small for states with angular momentum $l \approx 0$. In this case the effective expansion parameter turns out not to be α_2 (which is true for extremely large l) but $X^2\alpha_2$,

A	136	180	184	212	264	276	312	372	392
I/I_{hydr}	10.5	8.8	7.9	10.4	9.4	5.0	6.5	6	15.6

where $X \equiv k_0R_0$, and $X^2 \approx 100$. Therefore the range of applicability of perturbation theory for the moment of inertia is determined, strictly speaking, by $X^2\alpha_2$ since the states with $l = 0; 1; \ldots$ give a large contribution. However, one can expect that the averaged value of the moment will be correct over a substantially larger range of α_2. This can be explained as follows. We divide all states into two groups, states with large l and states with small l, where the latter contain the cases $l = 0; 1; \ldots$. Both groups will give approximately equal but opposite contributions to the moment of inertia, with an appreciable mutual cancellation of terms containing the parameter $(\alpha_2X^2)^2$ taking place. This cancellation, incidentally, turns out to be incomplete. This explains the obtained oscillations of the moment of inertia as a function of A.

[1] D. Inglis, Phys. Rev. **96**, 1059 (1954).
[2] D. Inglis, Phys. Rev. **103**, 1786 (1956).
[3] A. Bohr and B. Mottelson, Kgl. Dansk. Vidensk. Selsk. Mat.-Fys. Medd, **30**, No. 1 (1955).

Translated by M. Danos
32

SOVIET PHYSICS JETP VOLUME 35 (8), NUMBER 1 JANUARY, 1959

ON THE POLARIZATION OF RECOIL NUCLEONS IN THE PHOTOPRODUCTION OF PIONS

N. F. NELIPA and V. A. FEOKTISTOV

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor February 24, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **35**, 244-248 (July, 1958)

A general expression for the polarization of recoil nucleons appearing during the production of pions by photons has been obtained on the basis of the momentum and parity conservation laws. As an example, an expression is derived for the polarization in pion production in s, p, and d states.

1. Measurement of the polarization of the recoil nucleons that appear in photoproduction of mesons would help clarify, in principle, many important problems connected with the difference between the Fermi and Yang solutions, with the determination of small phase shifts in meson-nucleon scattering, with the elimination of the Minami ambiguity, etc.

A theoretical study of the problems of polarization can be made, roughly speaking, in two ways. The first,[1] based on the use of the density matrix, leads to the most general expressions for polarization, a particular case of interest to us being the expression for the polarization P of recoil nuclei in meson photoproduction. The second method, which employs the phenomenological scattering S matrix,[2] is simpler, albeit more limited. An expression for P, was obtained by the last method in reference 3.*

In the present work we should like to call attention to still another possibility of obtaining a general expression for P within the framework of the S matrix. Unlike the authors of reference 3, we obtained a more general expression for P, in which summation over the spin projections of the initial particles leads to the Racah coefficient.† The use of such a formula facilitates the calcula-

*An expression for P, in the particular case when the mesons are produced only in the s and p states, is given by Fel'd[4] without proof.

†Naturally, the expression we obtained for P is the same as obtained with the aid of the density matrix.