ALPHA DECAY OF NONSPHERICAL NUCLEI

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We consider the system of equations for the radial functions describing the motion of alpha particles emitted in the α decay of nonspherical nuclei of arbitrary spin. We obtain equations which determine the boundary conditions for the radial functions on the nuclear surface, and formulas expressing the α-decay probability in terms of the values of the radial functions at the nuclear surface and the shape of the nucleus. A simple approximate formula is found for the dependence of the α-decay probability on the angular momentum l carried off by the α particle and the energy of the level in the daughter nucleus.

Various methods for approximate solution of the system of equations for the radial functions are analyzed for even nuclei (spin 0). It is shown that the terms that give rise to a coupling of the equations (due to the nonspherical part of the Coulomb interaction between the α particle and the nucleus) cannot be treated as a perturbation for $l > 2$.

An exact numerical solution of the system of equations is given for an elliptical nucleus, taking into account all multipole interactions, under the condition of constancy of the wave function on the nuclear surface. An estimate is made of the influence of higher harmonics in the expansion of the equation of the nuclear surface in Legendre polynomials. The computational results are compared with experiment.

1. INTRODUCTION

THE study of the intensity of α transitions to rotational levels enables us to obtain important information concerning the properties of nonspherical nuclei. The theory of the excitation of rotational levels in α decay has been treated by Rasmussen and Segal,[1] Strutinskii,[2,3] Nosov,[4,5] and Fröman.[6] However, comparison of the results of the theoretical computations with experiment shows systematic disagreements: the results differ by a factor of 1 to 10 for the 4^+ levels of even-even nuclei, and by a factor of several hundred for the 6^+ levels of these nuclei. (So far it has not been possible to reduce the theoretical formulas for odd nuclei to a form suitable for practical computation, which would allow one to compare them with experiment.) The published theoretical papers contain a variety of simplifications: neglect of certain terms in the expression for the kinetic energy of the nuclear top,[1] neglect of 2^4-pole (in all the published papers) and sometimes even of the quadrupole[2] interaction between the nucleus and the α particle, etc. The effect of such omissions has not been properly appreciated, even though the results for large angular momenta l turn out to be extremely sensitive to such simplifications.

Rasmussen and Segal[1] give the results of numerical computations of intensity of α decay. These computations, however, not only contain various simplifications which are not completely justified but they are also done for certain special values of the eccentricity which were selected on the basis of extrapolation of experimental data for the rare earths. It is therefore not clear whether one could choose the eccentricity in such a way as to explain the intensity of the decay to all known levels.

The computations which are presented below had three purposes. The first of these was to bring the formulas describing the α decay of nonspherical nuclei of arbitrary spin to a form suitable for practical calculations. In addition, we wanted to explain the applicability of the usual methods of computation, in which the quadrupole moment of the electrical interaction of the α particle and the nucleus (as well as the higher multipole moments) are treated as a perturbation. Anticipating our results, we may say that for $l \geq 4$ (where l is the angular momentum carried off by the α particle) this treatment proved to be impossible. Finally, we wanted to find out whether the disagreement between experiment and computations which we mentioned above is the result of the approximations and omissions of the particular papers themselves, or whether it is related to the approximate nature of the fundamental starting points of the whole theory: the assumption of the constancy of the α particle wave function on the boundary of the nucleus and the assumption of a simple shape for the nucleus (i.e., the assumption that in the expansion of the radius vector $R(\theta)$ to the nuclear surface in Legendre polynomials one can drop all terms except the first two or three.

2. GENERAL THEORY OF ALPHA DECAY

The fundamental equations describing the α decay of nonspherical nuclei of arbitrary spin have already been obtained in references 1 and 6. The formulas which we shall find for the boundary conditions at the nucleus make these equations suitable for practical computations. In our calculations we shall limit ourselves to the simplest case of "favored" α transitions, i.e., transitions in which the parity of the state and the projection of the angular momentum on the nuclear axis do not change during the α-decay process.

If the angular momentum I_0 of the decaying nucleus is different from zero, the probability of excitation in α decay of the rotational level of the daughter nucleus having angular momentum I ($I = I_0,\ I_0 + 1,\ I_0 + 2 \ldots$ etc.) is given by

$$W_I = \frac{\hbar k_I}{\mu} \sum_{l=|I-I_0|}^{I+I_0}{}' |a_{Il}|^2. \tag{2.1}$$

The sum in (2.1) extends only over even values of l (we shall indicate such sums by a prime on the summation sign).

The quantities a_{Il} are the amplitudes at infinity of the radial functions $f_{Il}(r)$, describing α particles carrying off angular momentum l (and the residual daughter nucleus in a state with angular momentum I):

$$f_{Il}(r) \to a_{Il} \exp\{i(k_I r - \eta_I \ln 2k_I r)\}, \quad r \to \infty, \tag{2.2}$$
$$k_I^2 = 2\mu E_I / \hbar^2, \quad \eta_I = 2Ze^2\mu / \hbar^2 k_I,$$

where E_I is the α-decay energy in the transition to the level I. We shall denote the corresponding quantities for the transition to the ground level with $I = I_0$ simply as k, η, E.

The system of equations[1,2] satisfied by the radial functions $f_{Il}(r)$ (cf. also Appendix) is

$$\frac{d^2 f_{Il}}{dr^2} + \left(k_I^2 - \frac{l(l+1)}{r^2} - \frac{2k_I\eta_I}{r}\right) f_{Il}(r) = \sum_{l'=0}^{\infty}{}' \sum_{I'=|l'-I_0|}^{l'+I_0} V_{Il;I'l'}(r) f_{I'l'}(r). \tag{2.3}$$

The quantities $V_{Il;I'l'}(r)$ are the matrix elements of the noncentral part of the coulomb interaction of the α particle and the nonspherical nucleus, which can be represented as

$$V_{Il;I'l'}(r) = \sum_{L=2}^{\infty}{}' Q_L(Il;\ I'l') V_L(r);$$
$$Q_L(Il;\ I'l') = (-1)^{I-I'} \sqrt{(2l'+1)(2I'+1)} \times W(I'I_0Ll;\ l'I)\, C^{l0}_{L0;\ l'0}\, C^{II_0}_{I'I_0;\ L0}; \tag{2.4}$$
$$V_L(r) = \frac{2\mu}{\hbar^2}\frac{2L+1}{2} \int_{-1}^{1} V(r, \theta) P_L(\cos\theta)\, d(\cos\theta),$$

where W is a Racah coefficient,[7] the quantities $C^{IM}_{I_1M_1;I_2M_2}$ are Clebsch-Gordan coefficients, and the $V_L(r)$ are the coefficients in the Legendre polynomial expansion of the coulomb energy $V(r, \theta)$ of the α particle in the field of the nonspherical nucleus (θ is the polar angle in the coordinate system fixed in the nucleus.

The system (2.3) should be solved under some boundary condition on the nuclear surface $r = R(\theta)$. At the nuclear surface the wave function $\Psi(r, \theta, \varphi)$, which describes the motion of the α particle formed in the decay, depends only on the angle θ:

$$\Psi(R(\theta), \theta, \varphi) \equiv \chi(\theta)$$

(where one usually assumes that $\chi(\theta) = \text{const.}$). It can be shown (cf. Appendix) that this condition gives the system of equations

$$\sum_{I,l} C^{II_0}_{I_0,I_0+\Omega;\ l,-\Omega} f_{Il}[R(\theta)]\, Y_{l\Omega}(\theta, \varphi) = R_0(\theta)\,\chi(\theta)\,\delta_{\Omega,0}, \tag{2.5}$$

where Ω (the projection of the α particle angular momentum on the nuclear axis) takes on all possible values.

The boundary conditions (2.5) (for given $R(\theta)$ and $\chi(\theta)$) together with Eqs. (2.3) uniquely determine the functions $f_{Il}(r)$, and consequently via (2.2) give the amplitudes a_{Il}.

Making use of (2.5) we can obtain the approximate formula for the α-decay intensity which we have used previously.[8] Let us replace the functions f_{Il} by functions $\varphi_{Il}(r)$ which, for $r \to \infty$, go over into $\exp\{i(k_I r - \eta_I \ln 2k_I r)\}$, so that

$$\varphi_{Il}(r) = f_{Il}(r) / a_{Il}. \tag{2.6}$$

We multiply both sides of (2.5) by

$$C^{II_0}_{I_0,I_0+\Omega;\ l,-\Omega} Y^*_{l\Omega}(\theta, \varphi) / \varphi_{Il}[R(\theta)],$$

sum over Ω and integrate over θ and φ. Using the orthogonality property of the Clebsch-Gordan coefficients and the orthogonality of the spherical harmonics, we get

$$a_{Il} + \sum_{I'l'}{}'' B(I, l;\ I'l')\, a_{I'l'} = A_{Il} C^{II_0}_{I_0 l_0;\ l0} \tag{2.7}$$

(the summation is extended over all values of I' and l' except the one pair for which $I = I'$ and $l = l'$). In (2.7) we have introduced the notation

$$A_{Il} = 2\pi \int_{-1}^{1} \frac{R(\theta)\chi(\theta)}{\varphi_{Il}[R(\theta)]} Y_{l0}(\theta)\, d(\cos\theta), \tag{2.8}$$

$$B(I, l;\ I', l') = \sum_{\Omega} C^{II_0}_{I_0, I_0+\Omega;\ l, -\Omega} C^{I'I_0}_{I_0, I_0+\Omega;\ l', -\Omega} \times \int Y^*_{l\Omega} \frac{\varphi_{I'l'}[R(\theta)]}{\varphi_{Il}[R(\theta)]} Y_{l'\Omega}\, d\varphi\, d(\cos\theta).$$

As numerical estimates show, all the quantities $B(Il;\ I'l')$ are small, so that to a good approximation the whole summation in (2.7) can be dropped. This comes about because the functions $Y_{l\Omega}$ with different l are orthogonal, while the ratio $\varphi_{I'l'}[R(\theta)]/\varphi_{Il}[R(\theta)]$ varies only little with angle θ.* Neglecting the dependence of this ratio on angle, we find that for $l \neq l'$ the integral appearing in $B(Il;I'l')$ vanishes while for $l = l'$, $B(Il;I'l')$ vanishes because of the orthogonality of the Clebsch-Gordan coefficients with $I \neq I'$. We therefore get

$$a_{Il} \approx A_{Il} C^{II_0}_{I_0 l_0;\ l0}. \tag{2.9}$$

Now let us consider the quantities A_{Il} defined in (2.8). The functions $\varphi_{Il}(r)$ in the denominator depend on r essentially in the same way as the Coulomb functions $\varphi^{(0)}_{Il}(r)$ which are the exact solution of (2.3) when we neglect the right hand side (and satisfy the same boundary condition at $r \to \infty$,

$$\varphi^{(0)}_{Il}(r) \to \exp i\,(k_I r - \eta_I \ln 2k_I r),$$

as the functions $\varphi_{Il}(r)$). At the nucleus, this function falls off exponentially with increasing r, so that the principal contribution to the integral (2.8) comes from the neighborhood of $\theta = 0$, where $R(\theta)$ reaches a maximum. Also making use of the fact that all the functions $\varphi_{Il}[R(\theta)]$ depend on angle in approximately the same way, we find

$$A_{Il} \approx C\sqrt{2l+1}\,/\,\varphi_{Il}[R(0)], \tag{2.10}$$

where $\sqrt{2l+1} = \sqrt{4\pi}\,[Y_{l_0}(\theta)]_{\theta=0}$. In α decay, the constant η_I in (2.3) is always large compared to unity: $\eta_I \gg 1$, which is the reason why the quasi-classical approximation is well justified. In this approximation it is well known (also cf. the next section) that $\varphi_{Il}(r)$ is representable in the form

$$\varphi_{Il}(r) = \exp\{-S_l(r, k_I)\},$$

where S_l can be expanded in a rapidly converging series

$$S_l = A(r, k_I) + B(r, k_I)\, l(l+1) + C(r, k_I)\, l^2(l+1)^2 \tag{2.11}$$

and

$$k_I = k\sqrt{1 - \Delta E_I/E} \approx k - k\Delta E_I/2E,$$

$$\Delta E_I = \frac{\hbar^2}{2\mathcal{J}}\{I(I+1) - I_0(I_0+1) + a[(-1)^{I+1/2}(I+1/2)+1]\,\delta_{I_0, 1/2}\} \tag{2.12}$$

(E is the energy of the α-decay to the lowest level of the rotational band, which has spin I_0).

Expanding the coefficients of the series (2.11) in powers of $\Delta E_I/E$, we find that the function $\varphi_{Il}[R=0] = \varphi_{Il}(a)$ can be represented as the rapidly converging series

$$\varphi_{Il}(a) = c_1 \exp\left\{-\frac{\alpha}{2}\, l(l+1) - \frac{\beta}{2}\frac{\Delta E_I}{E} + \ldots\right\}. \tag{2.13}$$

For estimating orders of magnitude, it is useful to expand the Coulomb function $\varphi_{Il}^{(0)}(a)$, which is of the same order of magnitude as $\varphi_{Il}(a)$, in a series of the type of (2.13). As is easily verified [cf. formulas (3.11) and (3.12)],

$$\varphi^{(0)}_{Il}(a) = c_2 \exp\left\{-\frac{\alpha_c}{2}\, l(l+1) - \frac{\beta_c}{2}\frac{\Delta E_I}{E}\right\}, \tag{2.14}$$

$$\alpha_c = \tan\varepsilon/\eta \approx 0.1,\quad \beta_c = 2\eta(\varepsilon + 1/2\sin 2\varepsilon) \approx 75,$$

$$\varepsilon = \arccos\sqrt{ka/2\eta}.$$

Substituting (2.9), (2.10), and (2.13) in (2.1), we get finally

$$W_I = W_0 \sum_{l=I-I_0}^{I+I_0}{}' |C^{II_0}_{I_0 l_0;\ l0}|^2 (2l+1) \exp\left\{-\alpha l(l+1) - \beta\frac{\Delta E_I}{E}\right\}. \tag{2.15}$$

The coefficients α and β in (2.15) should be regarded as parameters whose values are determined from experiment, since the connection with the shape of the nucleus was lost in getting equation (2.10) from (2.8).

We note, finally, that for the case of even-even nuclei, when $I_0 = 0$, $I = l$ and $\Delta E_I/E \sim l(l+1)$, formula (2.15) assumes a simple form, given by Landau:[9]

$$W_l = (2l+1)\, W_0 e^{-\alpha' l(l+1)}. \tag{2.16}$$

Let us make one more remark concerning the derivation of formula (2.15). The one assumption in the derivation which was not completely justi-

*Each of the quantities $\varphi_{Il}[R(\theta)]$ has a strong angular dependence, but their ratio depends only slightly on angle. For the functions $\varphi^{(0)}_{Il}[R(\theta)]$, which are the solutions of (2.3) with the right side set equal to zero, this can be seen from the formulas of the next section. As the numerical computations show, this assertion is also valid for the functions $\varphi_{Il}[R(\theta)]$.

TABLE I. Alpha Decay of Some Even Nuclei*

Decaying Nucleus	α'	Level spin and parity	Particle energy kev	Level energy kev	Intensity of α decay (%) Experiment	Computation
Pu^{240}	0.457	0+	5158.9	0	75.5	75.5
		2+	5114.4	45.3	24.4	24.4
		4+	5014	147	$9.1\cdot10^{-2}$	$7.3\cdot10^{-2}$
Pu^{238}	0.435	0+	5491	0	71.1	71.1
		2+	5448	43.7	28.7	28.7
		4+	5352	141.5	0.13	0.11
U^{232}	0.394	0+	5318	68	68	68
		2+	5261	32	32	32
		4+	5134	0.3	0.3	0.23
Cm^{242}	0.441	0+	6110	0	73.7	73.7
		2+	6066	44	26.3	26.3
		4+	5965	148	$3.5\cdot10^{-2}$	$9.9\cdot10^{-2}$

*Formula (2.16) was used for the computations. The probability of α decay to the 2^+ level was used for the determination of the parameter α'.

TABLE II. Alpha Decay of Some Odd Nuclei*

Decaying Nucleus	α	β	Level spin and parity	Particle energy kev	Level energy kev	Intensity of α decay (%) Experiment	Computation
Pu^{239}	0.309	115	1/2±	5147	0	72	72
			3/2±	5134	13.2	16.8	16.8
			5/2±	5096	51.7	10.7	10.8
			7/2±	5064	84	$3.7\cdot10^{-2}$	$9\cdot10^{-2}$
			9/2±	4991	151	$1.3\cdot10^{-2}$	$2.6\cdot10^{-2}$.
Am^{241}	0.283	99	5/2±	5482	0	85	85
			7/2±	5439.1	43.4	12.8	12.8
			9/2±	5386.0	97.4	1.66	1.7
			11/2±	5321	164	$1.5\cdot10^{-2}$	$3.4\cdot10^{-2}$
			13/2±	5241	245	$2\cdot10^{-3}$	$2.4\cdot10^{-3}$

*Formula (2.15) was used for the computations. The probabilities of α decay to the levels with spin $I_0 + 1$ and $I_0 + 2$ were used for determining the constants α and β.

fied was that the integral (2.8) can be replaced by a quantity proportional to the value of the integrand at the point $\theta = 0$. However, the form of (2.15) does not depend in any essential way on this assumption, since (2.15) also follows from (2.8) and (2.9) if we expand the whole integral (2.8) in series:

$$\int_{-1}^{1} \frac{R(\theta)\chi(\theta)}{\varphi_{Il}[R(\theta)]} Y_{l0}(\theta)\, d(\cos\theta)$$

$$= (2l+1)\exp\Big[-\frac{1}{2}\Big\{c + \alpha l(l+1) + \beta\frac{\Delta E_I}{E} \quad (2.17)$$

$$+ \alpha_1 l^2(l+1)^2 + \beta_1\Big(\frac{\Delta E_I}{E}\Big)^2 + \gamma l(l+1)\frac{\Delta E_I}{E} + \ldots\Big\}\Big].$$

It is clear from this that formula (2.15) is quite exact. But this precision is obtained at the expense of giving up the possibility of theoretical evaluation of the constants α and β starting from the nuclear shape. Thus formula (2.15) cannot be used for determining the shape of α-active nuclei. It is not difficult to obtain an approximate formula of the same type as (2.15) but without this defect. To do this we need only replace $\varphi_{Il}[R(\theta)]$ in (2.8) by the Coulomb function $\varphi^{(0)}_{Il}[R(\theta)]$. From (2.1), (2.8) and (2.9) we then find

$$W_I = 4\pi^2\frac{\hbar k_I}{\mu}\sum_{l=I-I_0}^{I+I_0}{}' |C^{II_0}_{I_0 l;\, l0}|^2 \left|\int_{-1}^{1}\frac{R(\theta)\chi(\theta)}{\varphi^{(0)}_{Il}[R(\theta)]}Y_{l0}(\theta)\, d(\cos\theta)\right|^2. \quad (2.18)$$

The integral in this formula can be computed numerically for any nuclear shape $R(\theta)$, as soon as we choose the form of the function $\chi(\theta)$ (usually we assume that $\chi(\theta) = \text{const}$). Formula (2.18) cannot of course pretend to be particularly exact, in particular for large values of l, since at $l = 4$ (cf. the following section) the effect of the right-hand side of (2.3) already produces a sizeable difference getween the true radial functions $\varphi_{Il}(r)$ and the Coulomb functions $\varphi^{(0)}_{Il}(r)$, which are the solution of (2.3) without the right-hand side.

In conclusion we give some numerical data illustrating the applicability of formulas (2.15) and (2.16) to the description of the experimental data. As we see from Table I, Landau's formula (2.16) with the single parameter α' gives a good description of the experimental data for $l \leq 4$. For $l = 6$, this formula already leads to a marked disagreement with experiment (by a factor of hundreds!), which shows the need for considering the next term, $\alpha'' l^2 (l+1)^2$, in the expansion of the exponential. It is therefore natural to expect that formula (2.15) will give a good description of the intensities of α decay to the first five levels of the rotational band (for which $l \leq 4$), as is shown in Table II.

As we see from formula (2.8), the intensity of the α decay to particular levels can take on anomalously large or small values if some particular term in the Legendre polynomial expansion of $R(\theta)\chi(\theta)/\varphi_{ll}[R(\theta)]$ is "resonantly" large or small. The smooth representation (2.17) is not possible in such a case. This may be the explanation of the large (factor of three) discrepancy between the calculated and experimental amplitudes for the α decay of Cm^{242} to the 4^+ level of the daughter nucleus (cf. Table I).

3. ANALYSIS OF SOME APPROXIMATE METHODS OF SOLUTION OF EQUATIONS (2.3)

The system of equations (2.3) cannot be solved exactly by analytic methods even in the simplest case of $I_0 = 0$. Various authors have either completely omitted the right hand side of (2.3) or have regarded it as a perturbation and taken it into account by a method of successive approximations. In doing this, only the quadrupole part of the Coulomb interaction was included. We shall show that such a procedure for solution of the problem is incorrect, since the correction provided by the first approximation is of the same order of magnitude de as the zeroth approximation.

For the case of $I_0 = 0$, the equations for the radial functions and the appropriate boundary conditions have a very simple form, which can be obtained by setting $I_0 = 0$, $I = l$, $I' = l'$ in (2.3), (2.4), and (2.5). In this case

$$Q_L(l, l, l', l') = \sqrt{\frac{2l'+1}{2l+1}}\,|C^{l0}_{L0;\,l'0}|^2 = \int Y^*_{l'0} P_L Y_{l0}\, d\Omega.$$

Introducing the dimensionless variable

$$x = r/r_0, \quad r_0 = 2Ze^2/E = 2\eta/k$$

(where r_0 is the barrier radius) into (2.3), and setting

$$f_{ll}(r) = a_l \varphi_l(r), \quad \varphi_l(r) \to \exp i(k_l r - \eta_l \ln 2k_l r) \text{ for } r \to \infty,$$

we get

$$\varphi_l''(x) + (\varkappa_l^2 - \nu_l)\varphi_l(x) = \frac{4\eta^2}{x} \sum_{l'}{}' \frac{a_{l'}}{a_l} \varphi_{l'}(x) U_{l'l}(x). \quad (3.2)$$

Here we have introduced the notation

$$\varkappa_l^2 = 4\eta^2\left[1 - \frac{\Delta E}{6E} l(l+1)\right]; \quad \nu_l = 4\eta^2\left[\frac{1}{x} + \frac{l(l+1)}{4\eta^2 x^2}\right]. \quad (3.3)$$

ΔE is the excitation energy of the first rotational level,

$$U_{l'l} = \int Y_{l'0}(\mu) U(x, \mu) Y_{l0}(\mu)\, d\Omega; \quad \mu = \cos\theta, \quad (3.4)$$

and U is the noncentral part of the Coulomb interaction, which is given by

$$V(r, \theta) = \frac{E}{x}[1 + U(x, \mu)]. \quad (3.5)$$

The dimensionless constant η (cf. text following Eq. 2.12) is approximately equal to 25.

The solution of Eq. (3.2) without the right hand side is the coulomb function $\varphi_l^{(0)}(x)$, which can be represented quite accurately* as

$$\varphi_l^{(0)}(x) = \sqrt{\frac{\pi}{2}}\, e^{2\pi i/3} \left(\frac{\varkappa_l^2}{\varkappa_l^2 - \nu_l}\right)^{1/4} \times \left(\int_{x_l}^{x} \sqrt{\varkappa_l^2 - \nu_l}\right)^{1/2} H^{(1)}_{1/3}\left(\int_{x_l}^{x} \sqrt{\varkappa_l^2 - \nu_l}\, dx\right), \quad (3.6)$$

where $H^{(1)}_{1/3}$ is the Hankel function of the first kind of order $1/3$, and x_l is the turning point (the value at which $\kappa_l^2 = \nu_l$). The function (3.6) coincides with the exact solution (3.2) in the region of the turning point, and has the correct asymptotic behavior far away from that point. The constant in (3.6) is chosen so that all the functions $\varphi_l^{(0)}$ are real at and near the nucleus.

The goal of the computation is to calculate the amplitudes $a_{ll} = a_l$, which determine the α-decay probability W_l through (2.1). According to (2.8) and (2.9),

$$a_l = 2\pi \int_{-1}^{1} \frac{R(\theta)\chi(\theta)}{\varphi_l[R(\theta)]} Y_{l0}(\theta)\, d(\cos\theta). \quad (3.7)$$

We obtain the zeroth approximation $a_l^{(0)}$ to the value of a_l by replacing $\varphi_l[R(\theta)]$ in (3.7) by the Coulomb function $\varphi_l^{(0)}[R(\theta)]$, in accordance with (2.18). To find $a_l^{(1)}$ we must substitute the first approximation $\varphi_l^{(1)}$ in (3.7). In calculating $\varphi_l^{(1)}$, the functions $\varphi_l^{(0)}(x)$ and the values $a_l^{(0)}$ and $a_{l'}^{(0)}$ were substituted on the right side

*Formula (3.6) describes the variation of the function $\varphi_l^{(0)}$ in the neighborhood of the nucleus and its dependence on l with an accuracy which is no worse than 1%.

of (3.2). It is then convenient to look for a solution of (3.2) in the form $\varphi_l^{(1)}(x) = \lambda_l(x)\varphi_l^{(0)}(x)$. We then find for λ_l the equation

$$2\lambda_l'\varphi_l'^{(0)}(x) = \frac{4\eta^2}{x}\sum_{l'}\frac{a_{l'}^{(0)}}{a_l^{(0)}}U_{l'l}\varphi_{l'}^{(0)}(x)$$

with the boundary condition $\lambda_l(\infty) = 1$. (On the left hand side we have neglected the term $\lambda_l''(x)\varphi_l^{(0)}(x)$, which is vanishingly small compared to $\lambda_l'(x)\varphi_l'^{(0)}(x)$. This is obvious simply from the fact that in the region below the barrier the function $\varphi_l^{(0)}(x)$ changes by 20 orders of magnitude whereas $\lambda_l(x)$ only changes by a factor of 1 to 10.)

Performing the integration, we get

$$\lambda_l(x) = 1 + \eta\sum_{l'}\frac{a_{l'}^{(0)}}{a_l^{(0)}}\int_x^\infty \frac{\varphi_{l'}^{(0)}(x)}{\varphi_l^{(0)}(x)}\frac{U_{l'l}(x)}{\sqrt{x(1-x)}}dx. \quad (3.8)$$

According to (3.7),

$$a_l^{(1)} = a_l^{(0)}/\lambda_l(a), \quad a = [R(\theta)]_{\theta=0}/r_0 \quad (3.9)$$

(a is the nuclear radius in units of r_0).

For practical computations it is convenient to use the asymptotic representation (3.6) which gives a quite accurate description of the behavior of $\varphi_l^{(0)}$ near the nuclear surface:

$$\varphi_l^{(0)}(x) \approx \left(\frac{\varkappa_l^2}{\nu_l - \varkappa_l^2}\right)^{1/4}\exp\left\{\int_x^{x_l}\sqrt{\nu_l - \varkappa_l^2}\,dx\right\}, \quad \frac{x}{x_l} \ll 1. \quad (3.10)$$

The exponent is well approximated by the series

$$\int_x^{x_l}\sqrt{\nu_l - \varkappa_l^2}\,dx = A + Bl(l+1) + Cl^2(l+1)^2, \quad (3.11)$$

where

$$A(x) = 2\eta(\alpha - {}^1/_2\sin 2\alpha),$$
$$B(x) = 2\eta\left[\frac{\Delta E}{12E}\left(\alpha + \frac{1}{2}\sin 2\alpha\right) + \frac{\tan\alpha}{4\eta^2}\right],$$
$$C(x) = \frac{1}{2\eta}\frac{\Delta E}{6E}\cot\alpha + \eta\left(\frac{\Delta E}{6E}\right)^2\left(\cot\alpha + \frac{3}{2}\alpha + \frac{1}{4}\sin 2\alpha\right) + \frac{1}{3\eta^3}\left(\cot 2\alpha - \frac{1}{8\sin\alpha\cos^3\alpha}\right), \quad (3.12)$$
$$\alpha = \arccos\sqrt{x}.$$

(Formulas (3.10), (3.11), and (3.12) are also valid for nuclei with $I_0 \neq 0$. In this case $(\Delta E/6E)\, l(l+1)$ must be replaced by ΔE_I and κ_I by κ_l.)

We point out for orientation that at the nuclear surface $A \approx 40$, $B \approx 0.1$ and $C \approx 10^{-4}$, so that for small l the term $Cl^2(l+1)^2$ is unimportant. Substitution of (3.10) and (3.11) in (3.8) gives

$$\lambda_l(x) = 1 + \eta\sum_{l'=0}^\infty\frac{a_{l'}^{(0)}}{a_l^{(0)}}\int_x^1\frac{U_{l'l}(x)}{\sqrt{x(1-x)}} \quad (3.13)$$
$$\times\exp\{B(x)[l'(l'+1) - l(l+1)]\}\,dx.$$

In (3.13) the upper limit of integration has been replaced by 1. This step is necessary since the expansion (3.11) is not valid in the region $\nu_l - \kappa_l^2 < 0$. Because of the rapid falloff of $U_{l'l}$ with increasing x and the oscillatory character of the functions $\varphi_l^{(0)}$ in the region $x > 1$, the integral from 1 to ∞ contributes practically nothing.

For numerical computations it is necessary to choose a specific shape for the nucleus,

$$R(\theta) = x_1[1 + \alpha_2 P_2(\cos\theta) + \alpha_4 P_4(\cos\theta) + \ldots]. \quad (3.14)$$

For an ellipsoid with eccentricity $u = \sqrt{1 - b^2/a^2}$ and semiaxes a and b (in units of r_0),

$$\alpha_2 = \frac{15}{4u^2}\left[1 - \frac{2}{3}u^2 - \frac{u\sqrt{1-u^2}}{\arcsin u}\right] = \frac{u^2}{3} + 0.16u^4 + 0.098u^6 + \ldots, \quad (3.15)$$
$$\alpha_4 = \frac{27}{64u^2}[35 - 40u^2 + 8u^4] + \frac{45}{64}\frac{\sqrt{1-u^2}}{u^3\arcsin u}(10u^2 - 21) = 0.086u^4 + 0.083u^6 + \ldots,$$

The quantities $U_{l'l}$ appearing in (3.13) are given by the coefficients in the Legendre polynomial expansion of $U(x, \mu)$. In the region outside the nucleus (but not in the interior!) the potential of a uniformly charged ellipsoid is correctly given by the formula

$$U(x, \mu) = 3\sum_{n=1}^\infty\frac{1}{(2n+1)(2n+3)}\left(\frac{ua}{x}\right)^{2n}P_{2n}(x). \quad (3.16)$$

Formula (3.16) is not the same expansion as that usually used for the potential of an ellipsoid, and is considerably more convenient. (The possibility of having different expansions of the potential in Legendre polynomials in the region $b \le x \le a$ is related to the fact that when this inequality is satisfied the Legendre polynomials are not orthogonal in the region external to the nucleus.)

For an arbitrary nuclear shape (3.14), the expansion (3.16) is not exact. As we shall show later, the details of the shape of the nucleus have a significant effect on the probability of emission of particles with high l. This effect is, however, associated mainly with the change in the boundary conditions, whereas the effect of nuclear shape which is associated with changes in the potential is not so important and was not taken into account. In our computations we therefore assumed the nucleus to be elliptical and used formula (3.16). Then

$$U_{l'l} = 3\sum_{n=1}^\infty\frac{1}{(2n+1)(2n+3)}\left(\frac{ua}{x}\right)^{2n}\frac{\sqrt{(2l+1)(2l'+1)}}{4n+1}(C_{l'0;l0}^{2n,0})^2. \quad (3.17)$$

In Tables III and IV we give the results of computation of the amplitudes $a_l^{(0)}$ and the factors

TABLE III. Probability of α decay to successive rotational levels, computed in zeroth approximation for U^{236}

	Computational results							Experimental data
α_2	0.07	0.095	0.117	0.141	0.168	0.195	0.227	
u^2	0.20	0.25	0.30	0.35	0.40	0.45	0.50	
$a_2^{(0)}/a_0^{(0)}$	0.38	0.46	0.55	0.62	0.69	0.74	0.77	0.57
$a_4^{(0)}/a_0^{(0)}$	0.025	0.038	0.054	0.072	0.086	0.106	0.122	0.035
$a_6^{(0)}/a_0^{(0)}$	$4.7\cdot10^{-4}$	$9.1\cdot10^{-4}$	$1.6\cdot10^{-3}$	$2.4\cdot10^{-3}$	$3.3\cdot10^{-3}$	$4.4\cdot10^{-3}$	$5.8\cdot10^{-3}$	$6.7\cdot10^{-3}$

$\lambda_l^{(1)}(a)$, which determine the first order correction through Eq. (3.9). The computations were done for U^{236} (α decay of Pu^{240}). $\chi(\theta)$ was set equal to a constant in (3.7). The size of the nucleus was fixed by the condition $\chi^2(\theta)V_n = 1$, where V_n is the nuclear volume.

TABLE IV. First approximation factors $\lambda_l^{(1)}$ for U^{236}

u^2	0.25	0.30	0.35	0.40	0.45	0.50
$\lambda_0^{(1)}$	1.13	1.18	1.24	1.31	1.38	1.45
$\lambda_2^{(1)}$	1.22	1.23	1.25	1.26	1.27	1.29
$\lambda_4^{(1)}$	1.77	1.82	1.85	1.87	1.89	1.91
$\lambda_6^{(1)}$	2.13	2.21	2.31	2.42	2.51	2.56

As we see from Table IV, for actually occurring deformations ($u^2 = 0.3$), the correction factor λ is close to unity only for $l = 0$ and 2. Starting with $l = 4$ the correction reaches an order of magnitude of 100% and the method of successive approximations is not consistent.*

We tried to improve the successive approximation method by substituting $\varphi_l(x) = \lambda_l(x)\varphi_l^{(0)}(x)$ on the right as well as the left side of (3.2). When this is done one gets a system of coupled equations for the relatively slowly varying functions $\lambda_l(x)$. If one assumes that the $\lambda_l(x)$ vary significantly only near the nucleus, expands $\lambda_l(x)$ in powers of $(x-a)$ around the point $x = a$, and keeps only the zeroth and first powers, a relatively simple system of algebraic equations is obtained for $\lambda(a)$. The coefficients in these equations are integrals which can be computed numerically. However, the computation showed that the solutions obtained by this method diverge for actually occurring values of the deformation.

It should be mentioned that, aside from the inaccuracy associated with the use of the successive-approximation method, the computations involved the use of the approximate value (3.11) of the coulomb function and the approximate formula (3.7) which is a consequence of the boundary conditions on the nuclear surface. Estimates showed that both these approximations are very good, and do not contribute errors exceeding a few percent. Thus the divergence of the method of successive approximations is caused only by the use of the method itself. We were therefore faced with the necessity of an exact numerical solution of equations (3.2).

4. NUMERICAL SOLUTION OF THE EQUATIONS

The numerical solution of Eq. (3.2) expressed in terms of the functions $f_l(x) = a_l\varphi_l(x)$ was carried out on the M-2 electronic computer of the Laboratory for Control Machines and Systems of the Academy of Sciences, U.S.S.R. Seven functions ($l = 0, 2, \ldots 12$) were included in the computations, while the remainder were set equal to zero. The solution of the equations was carried "inward" from $x = \infty$ (actually from $x = 3$) to the surface of the nucleus (to $x \approx 0.1$). To determine the a_l, we constructed the 7 fundamental systems of solutions φ_{ln} ($n = 0, 2, \ldots 12$). The solution with number n was normalized at $x = 3$ by the requirement that $\varphi_{ln}(3) = \delta_{ln}\varphi_l^{(0)}(3)$. Every solution of (3.2) can be represented as a linear combination of these seven solutions. Thus each of the functions we are trying to find can be written as

$$f_l(x) = \sum_{n=0}^{12}{}' a_n \varphi_{ln}(x), \quad l = 0, 2 \ldots 12 \tag{4.1}$$

(where the prime on the summation sign means that the sum runs over even n). It is easy to see that the functions $f_l(x)$ have the correct asym-

*The values found for a_0 and a_2 can be used to determine the dimensions and eccentricity of the nucleus. However, when this is done there are no quantities left which can be compared with experiment in order to check the theory. We also mention that the divergence of the successive approximation method for $l \geq 4$ means that the excitation of higher rotational levels occurs to a considerable extent through the coulomb interaction of the α particle with the nucleus.

ptotic behavior for $x \to \infty$, so that the a_n are the required decay amplitudes. After substituting (4.1), the boundary condition (2.5) gives (in our case where $I_0 = 0$, $I = l$),

$$\sum_{n=0}^{12}{}' a_n \chi_n(\theta) = \chi(\theta), \tag{4.2}$$

where

$$\chi_n(\theta) = \sum_{l=0}^{12}{}' \frac{\varphi_{ln}[R(\theta)]}{R(\theta)} Y_{l0}(\theta) \tag{4.3}$$

are well-defined functions which are uniquely determined from the shape of the nucleus. The requirement

$$\chi(\theta) = \text{const} \tag{4.4}$$

enables us to find the ratio a_n/a_0.

As a check, the fundamental systems of solutions $\varphi_{ln}(x)$ were computed for $n = 0, 2, 4, 6, 8$, when only five functions f_0, f_2, f_4, f_6, f_8 were included. A comparison showed that then the first four functions are changed negligibly. The various stages of the solution can be followed on the graphs of Figs. 1 to 5.

To illustrate the orders of magnitude, Fig. 1 shows the real part of $\varphi_l^{(0)}(x)$ for U^{236} (α decay of Pu^{240}) as a function of x.

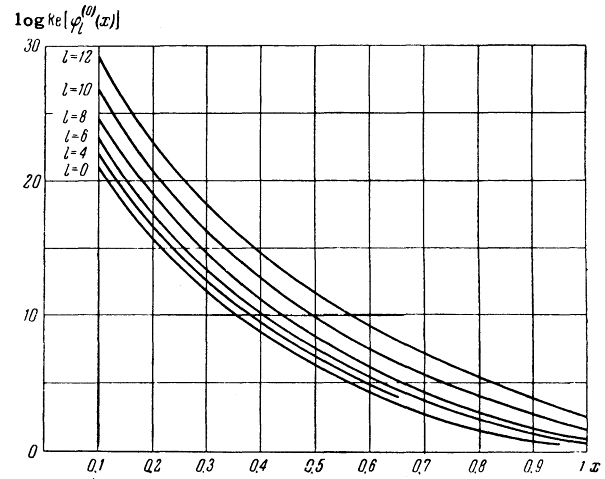

FIG. 1. Real part of the zeroth approximation functions $\varphi_l^{(0)}(x)$ for U^{236}; $\eta = 26.0$.

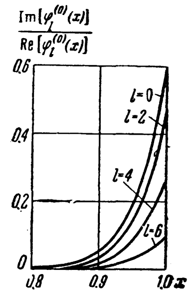

FIG. 2. Ratio of imaginary to real part of $\varphi_l^{(0)}(x)$.

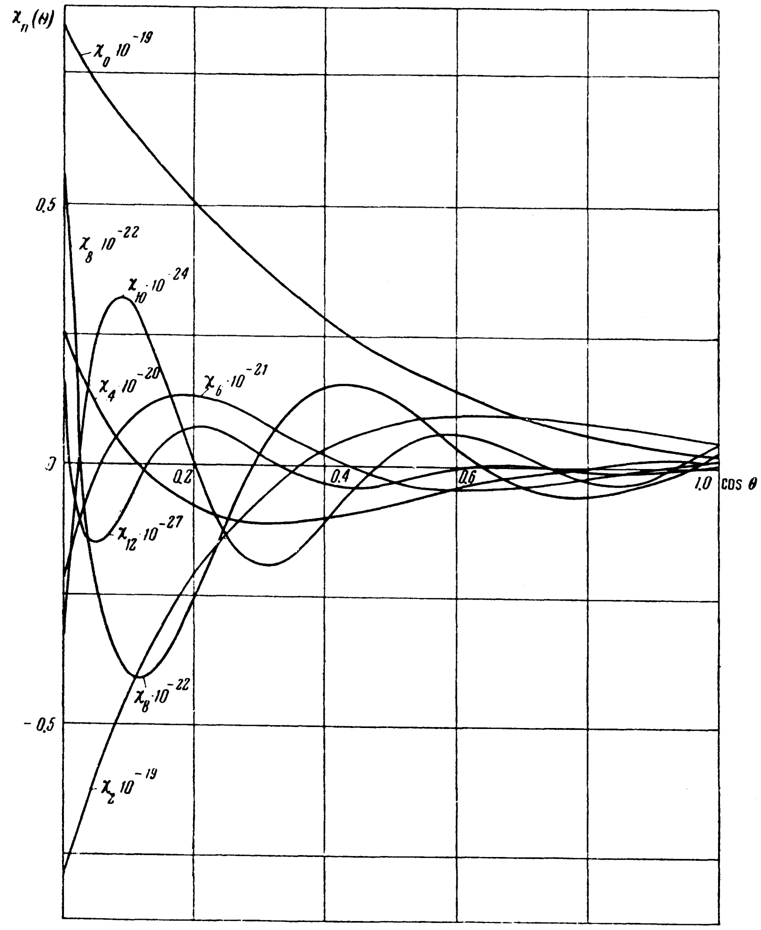

FIG. 3. Graph of the functions $\chi_n(\theta)$.

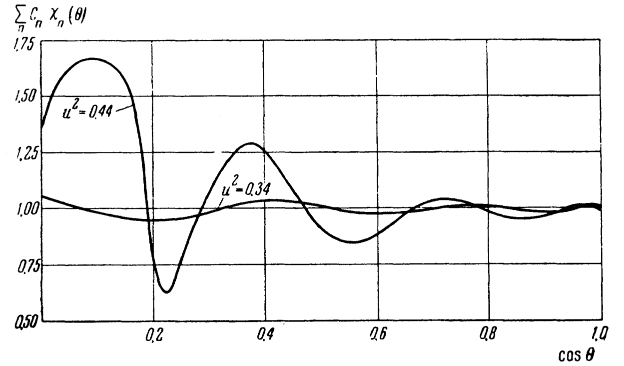

FIG. 4. Graph of the function $\sum_n c_n\chi_n(\theta)$. The coefficients c_n were selected by the least squares method so that the deviation of $\sum_n c_n\chi_n(\theta)$ from unity was a minimum. (The constants correspond to U^{236}).

Figure 2 shows the ratio of the imaginary part of $\varphi_l^{(0)}(x)$ to the real part. As we see from the figure, at $x = 0.8$ all the functions $\varphi_l^{(0)}(x)$ can already be assumed to be real, as should be the case according to (3.10). The functions $\varphi_{ln}(x)$ also turn out to be real at the nucleus. Choosing a real constant in (4.4), we also get real values for the amplitudes a_n.

Figure 3 shows graphs of the functions $\chi_n(\theta)$, calculated for $\eta = 25.4$, $a = 0.19$, $u^2 = 0.35$. The calculations were done keeping all the terms in the multipole expansion (3.16), (3.17) of the interaction potential. Linear combinations must be built up from the functions $\chi_n(\theta)$ so as to obtain a constant value on the nuclear surface, in accordance with (4.2) and (4.4). Each of the functions χ_n has $n/2$ roots on the nuclear surface. All of the functions increase rapidly as we move from the "nose" of the nucleus toward the equator, in accordance with the decrease of $R(\theta)$ in this direction. A practical choice of coefficients a_n satisfying the condition (4.4) was made by applying the least squares method to 17 points on the nucleus, at which the functions χ_n were calculated. In applying least squares, we used a weight factor of $1/\chi_0(\theta)$. The introduction of the weight factor enabled us to avoid excess sensitivity to the unimportant equatorial region of the nucleus. The large number of functions which were to combine to give the constant enabled us to fit it quite accurately. This is illustrated in Fig. 4 for the example of U^{236}. We see from this figure that the curve of the "deviations" intersects the value 1, to which the fit was made, a number of times which is one greater than the number of zeros in the highest function $\chi_n(\theta)$ included in the computation. We also note that the fit is quite good for $u^2 \leq 0.35$, while it is rather bad for $u^2 > 0.45$. Thus the

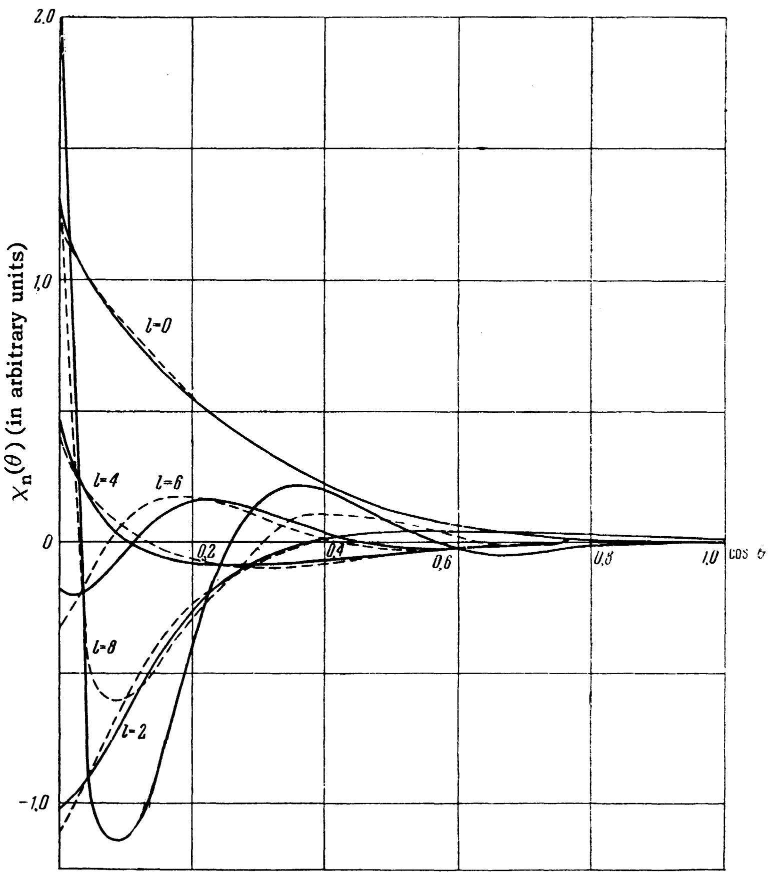

FIG. 5. Graph of the functions $\chi_n(\theta)$, calculated including five ($f_0, f_2, \ldots f_8$) or seven ($f_0, f_2, \ldots f_{12}$) functions f_n. The solid curves are for $l_{max} = 8$, the dotted curves for $l_{max} = 12$.

number of functions $\chi_n(\theta)$ included was insufficient for the latter case. This example illustrates especially clearly the need to include a large number of functions, and the inapplicability of the usual analytic methods. Figure 5 shows the functions $\chi_n(\theta)$ calculated including five $(f_0, f_2, \ldots f_8)$ and seven $(f_0, f_2, \ldots f_{12})$ functions f_n.

In our computational procedure we had first to assign the nuclear size (the computation was done for one volume and various eccentricities), so that it is important to examine the extent to which the results are sensitive to the size of the nucleus.

The results of such a computation are contained in Table V.

TABLE V. Effect of nuclear size on α-decay probability (for U^{236})

Major semi-axis a	Ratio of decay probabilities		
	a_2/a_0	a_4/a_0	a_6/a_0
0.156	0.56	0.062	0.0022
0.173	0.59	0.071	0.0027
0.190	0.62	0.080	0.0034

Comparing Table V with Table VI, which shows the effect of the nuclear shape, we easily discover that the nuclear size has a subordinate role in determining the relative intensities of α decays. In evaluating the results it should be remembered that a change in size of 10% (the extreme values of a in Table V differ by 10% from the middle value) gives a change of a factor of 5 in the absolute value of the function f_0, and consequently a change of a factor of 25 in the α-decay intensity.

TABLE VI. Intensity of α decay to rotational levels as a function of nuclear shape (for U^{236})

	Computational results				Experimental data
α_2	0.095	0.141	0.186	0.238	
u^2	0.25	0.35	0.44	0.52	
a_2/a_0	0.41	0.59	0.85	0.93	0.57
a_4/a_0	0.033	0.070	0.113	0.163	0.035
a_6/a_0	0.0011	0.0030	0.0072	0.011	0.0067

Table VI gives the results of numerical computation of α-decay intensities for U^{236} (α-decay of Pu^{240}) which were carried out on an electronic computer.

The data of Table VI are plotted in Fig. 6. The crosses on the curves are the experimental values.

Comparison of Table VI with Table III shows good agreement of the results of the exact calculations with those of the approximate calculations [using formula (2.18)]. As mentioned earlier, for

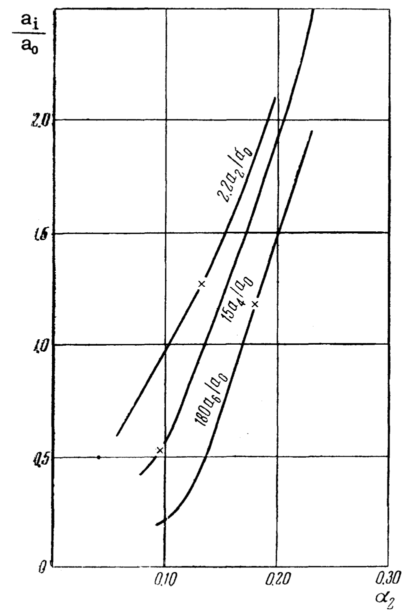

FIG. 6. Dependence of intensity of α decay to successive rotational levels on elongation of the nucleus (computation for U^{236}).

reasonable values of the eccentricity the first order corrections in the analytic computation are not small, so that the close agreement between the results of the numerical and approximate computations was unexpected. This leads one to think that applying (2.18) to odd nuclei with $I_0 \neq 0$ also should give good results and enable one to determine the eccentricity of the nucleus simply.

Such computations were done for Am^{241} and Pu^{239}. Only one parameter, the eccentricity u^2, was varied; the nuclear size was chosen to give the correct expression for the α-decay intensity to the ground state. The results of the computation are given in Table VII.

One striking feature of the results in Table VI is the clear, though not very large, discrepancy between the experimental and calculated values. Thus, by setting $\alpha_2 = 0.133$ we would obtain exact agreement of the theoretical and experimental values of a_2/a_0, but we then get a difference of a factor of 1.8 in the values of a_4/a_0 (and a discrepancy of a factor of 2.7 in the value of a_6/a_0). This discrepancy can be eliminated by a suitable choice of the coefficient α_4 in formula (3.14) which determines the nuclear shape. For arbitrary α_4 the nucleus is of course no longer ellipsoidal, and the matrix elements of the potential are determined by a formula which is different from (3.17), a point which was not taken into account in the computations. However, this selection of the coefficient α_4 does not improve the agreement with experiment for the ratio a_6/a_0 but rather makes it worse. This ratio, in turn,

TABLE VII. Intensity of α decay to rotational levels for some odd nuclei*

Daughter nucleus	Spin of rotational level	Relative intensity of α decay		
		Exptl.	Calculated for elliptical nucleus	Calculated for oval nucleus
Np^{237} $I_0 = 5/2$	7/2	0.150	0.150	0.146
	9/2	$2 \cdot 10^{-2}$	$2.8 \cdot 10^{-2}$	$2.5 \cdot 10^{-2}$
	11/2	$1.8 \cdot 10^{-4}$	$1.2 \cdot 10^{-3}$	$1.8 \cdot 10^{-4}$
	13/2	$2.4 \cdot 10^{-5}$	$1.2 \cdot 10^{-4}$	$1.8 \cdot 10^{-5}$
U^{235} $I_0 = 1/2$	3/2	0.232	0.232	0.232
	5/2	0.148	0.190	0.195
	7/2	$5 \cdot 10^{-4}$	$6 \cdot 10^{-3}$	$5.7 \cdot 10^{-4}$
	9/2	$1.4 \cdot 10^{-4}$	$2.7 \cdot 10^{-3}$	$2.6 \cdot 10^{-4}$

*The intensity was computed using formula (2.18). For N_p^{237}, we set $a = 0.193$ and $u^2 = 0.283$ in the case of the ellipse and $u^2 = 0.30$ and $\alpha_4 = -0.04$ in the case of the oval. For U^{235}, we used $a = 0.173$ and $u^2 = 0.285$ for the ellipse and $u^2 = 0.32$ and $\alpha_4 = -0.043$ for the oval. It is interesting to note that the two nuclei should be assigned almost exactly the same shape.

can be "fitted into place" by a suitable choice of α_6 in formula (3.14), but such a procedure can hardly be meaningful. For illustration, we give in Table VIII the results of the computation with this optimum choice of α_4.

We note that the quadrupole moment Q_0 calculated for an elliptical nucleus with $\alpha_2 = 0.121$, is equal to 10×10^{-24}, in beautiful agreement with the data found for this region of the periodic table from experiments on coulomb excitation.[10]

In conclusion, Table IX gives numerically computed intensities for some α transitions in even-even nuclei. In the computations the nucleus was assumed to be elliptical, and the eccentricity was chosen to give closest possible agreement with experiment for the value of a_2/a_0. A striking feature is the increase in α_2 as we move away from the closed shell corresponding to lead, the passage through a maximum and the subsequent decrease of α_2 as we go toward heavier nuclei.

APPENDIX

We give a short derivation of the system of equations (2.3) and the boundary condition (2.5).

TABLE VIII. Intensities of α decay to successive rotational levels, for elliptical ($\alpha_2 = 0.121$, $\alpha_4 = 0.011$) and oval ($\alpha_2 = 0.142$, $\alpha_4 = -0.029$) shape of the nucleus (U^{236})

	Experimental data	Elliptical nucleus	Oval nucleus
a_2/a_0	0.57	0.57	0.57
a_4/a_0	0.035	0.063	0.035
a_6/a_0	0.0067	0.0024	0.00015

The wave function ψ of the decaying nucleus corresponds to a definite value of the total angular momentum I_0 (the spin of the decaying nucleus) and its projection M on the z axis of a fixed coordinate system, and depends on the radius vector $\mathbf{r}(r, \theta_\alpha, \varphi_\alpha)$ to the position of the α-particle in the laboratory system and the Euler angles $\Theta_i = \Theta, \Phi, \Psi$ which determine the orientation of the elongated nucleus:

$$\psi = \psi_{I_0 M}(r, \theta_\alpha, \varphi_\alpha, \Theta_i). \tag{A.1}$$

We introduce a system of orthonormal functions $\Phi^{(Il)}_{I_0 MK}(\theta_\alpha, \varphi_\alpha, \Theta_i)$, which are eigenfunctions of:

TABLE IX. Intensities of α transitions in some even-even nuclei

Daughter nucleus	Decaying nucleus	u^2	α_2	a_2/a_0		$10\,(a_4/a_0)$		$10^3\,(a_6/a_0)$		$10^5\,(a_8/a_0)$	
				comp.	exp.	comp.	exp.	comp.	exp.	comp.	exp.
Cf^{250}	Fm^{254}	0.235	0.088	0.45	0.45	0.53	0.70	2.5		5	
Cm^{248}	Cf^{252}	0.226	0.094	0.41	0.41	0.40	0.50	1.4			
Cm^{242}	Cf^{246}	0.272	0.104	0.52	0.53	0.67	0.47	3	14	7	
Pu^{240}	Cm^{244}	0.291	0.113	0.52	0.55	0.60	0.13	3	8	5	
Pu^{238}	Cm^{242}	0.328	0.130	0.60	0.60	0.83	0.22	4.5	8	10	50
U^{238}	Pu^{242}	0.275	0.106	0.60	0.60	0.70		2.5		2	
U^{236}	Pu^{240}	0.330	0.121	0.57	0.57	0.63	0.35	2.5	6		
U^{234}	Pu^{238}	0.335	0.143	0.63	0.63	0.87	0.47	4	8	7	
Th^{226}	U^{230}	0.406	0.168	0.69	0.69	0.87	0.77	3		5	
Ra^{224}	Th^{228}	0.374	0.153	0.54	0.63	0.50	0.53	1		0.7	
Em^{218}	Ra^{222}	0.165	0.060	0.021	0.021	0.08		0.05			

(1) the angular momentum $\hat{l}$ of the α particle with eigenvalue $l(l+1)$ (in the fixed system); (2) the angular momentum of the nucleus, $\hat{I} = \hat{R} + \eta_\xi \hat{I}_0$, where R is the angular momentum of the rotation of the nucleus and η_ξ is a unit vector along the symmetry axis of the nucleus (eigenvalue $I(I+1)$); (3) the total angular momentum of the system $\hat{I}_0 = \hat{I} + \hat{l}$ (eigenvalue $I_0(I_0+1)$): (4) the projection $\hat{I}_{0z}$ of the vector $\hat{I}_0$ on the fixed z axis (value M); and (5) the projection $\hat{I}_\xi$ of the vector $\hat{I}$ on the symmetry axis η_ξ of the nucleus, $\hat{I}_\xi = \eta_\xi \cdot \hat{I}$ (value K). It is not hard to show that these functions have the form

$$\Phi^{(Il)}_{I_0MK}(\theta_\alpha, \varphi_\alpha, \Theta_i) = \sum_{m=-l}^{l} C^{I_0M}_{lm;\, I,\, M-m}\, Y_{lm}(\theta_\alpha, \varphi_\alpha)\, \widetilde{D}^{(I)}_{M-m,\,K}(\Theta_i). \tag{A.2}$$

Here the Y_{lm} are normalized spherical harmonics, while the $\widetilde{D}^{(I)}_{MK}$ are the coefficients in the representation of the rotation group, normalized so that

$$\int \widetilde{D}^{(I_2)*}_{M_2K_2} \widetilde{D}^{(I_1)}_{M_1K_1}\, d(\cos\Theta)\, d\Phi\, d\Psi = \delta_{I_1I_2}\delta_{M_1M_2}\delta_{K_1K_2},$$

where

$$\widetilde{D}^{(l)}_{0m}(\Theta, \Phi, \Psi) = \frac{1}{\sqrt{2\pi}} Y^*_{lm}\left(\Theta, \frac{\pi}{2} - \Phi\right);$$

$$\widetilde{D}^{(l)}_{m0}(\Theta, \Phi, \Psi) = \frac{1}{\sqrt{2\pi}} Y_{lm}\left(\Theta, \Psi - \frac{\pi}{2}\right); \tag{A.3}$$

$$Y_{lm}(\theta_\alpha, \varphi_\alpha) = \sum_{\Omega=-l}^{l} \sqrt{\frac{8\pi^2}{2l+1}}\, \widetilde{D}^{(l)}_{m\Omega}(\Theta, \Phi, \Psi)\, Y_{l\Omega}(\theta, \varphi),$$

and θ, φ are the polar angles of the vector $\mathbf{r}_\alpha$ in the coordinate system fixed in the nucleus.

Substituting (A.3) in (A.2) and using the identity

$$\sqrt{\frac{8\pi^2}{2l+1}} \sum_{m=-l}^{l} C^{I_0M}_{lm;\,I,\,M-m}\, \widetilde{D}^{(l)}_{m\Omega}(\Theta_i)\; D^{(I)}_{M-m,\,K}(\Theta_i) = \sqrt{\frac{2I+1}{2I_0+1}}\, C^{I_0,\,K+\Omega}_{l\Omega;\,IK}\, D^{(I_0)}_{M,\,K+\Omega}(\Theta_i),$$

we easily obtain $\Phi^{(Il)}_{I_0MK}$ an expression not in terms of θ_α, φ_α, as in (A.2), but in terms of the angles θ and φ in the coordinate system fixed in the nucleus:

$$\Phi^{(I,l)}_{I_0MK} = \sqrt{\frac{2I+1}{2I_0+1}} \sum_{\Omega=-l}^{l} C^{I_0,\,K+\Omega}_{l\Omega;\,IK}\, \widetilde{D}^{(I_0)}_{M;\,K+\Omega}\, Y_{l\Omega}(\theta, \varphi). \tag{A.4}$$

We represent the wave function (A.1) by a series in the functions $\Phi^{(Il)}_{I_0MK}$ defined by either (A.2) or (A.4). The expansion coefficients F_{IlK} will be functions of r:

$$\psi_{I_0M} = \sum_{IlK} F_{IlK}(r)\, \Phi^{(Il)}_{I_0MK}(\theta_\alpha, \varphi_\alpha, \Theta_i). \tag{A.5}$$

The function ψ_{I_0M} is a solution of the Schrodinger equation:

$$\left\{-\frac{\hbar^2}{2\mu}\nabla^2_\alpha + \frac{\hbar^2}{2\mathcal{J}}(\hat{R}^2_\xi + \hat{R}^2_\eta) + \frac{\hbar^2}{2\mathcal{J}'}\hat{R}^2_\zeta + V(r, \theta)\right\}\psi_{I_0M} = E\psi_{I_0M}. \tag{A.6}$$

$\mathcal{J}'$ is the moment of inertia with respect to the nuclear symmetry axis, and $\mathcal{J}$ with respect to the perpendicular axis. We note that

$$\hat{R}_\zeta \Phi^{(Il)}_{I_0MK} = (\hat{\mathbf{I}} - \hat{\eta}_\zeta \hat{I}_0)_\zeta\, \Phi^{(Il)}_{I_0MK} = (\hat{I}_\zeta - I_0)\,\Phi^{(Il)}_{I_0MK} = (K - I_0)\,\Phi^{(Il)}_{I_0MK}.$$

If $\mathcal{J}' \to 0$, which is the only case we shall treat, the terms involving $\hat{R}^2_\xi$ must be absent (since they give an infinite energy), from which it follows very easily that all terms with $K \neq I_0$ must be absent from (A.5). We should therefore set $F_{IlK} = r^{-1} f_{Il}(r)\,\delta_{KI_0}$, which gives

$$\psi_{I_0M} = \sum_{Il} \frac{f_{Il}(r)}{r}\, \Phi^{(Il)}_{I_0MI_0}(\theta_\alpha, \varphi_\alpha, \Theta_i). \tag{A.7}$$

Formula (2.1) follows quickly from equation (A.7).

Writing $\hat{R} = \hat{I} - \eta_\xi I_0$ and noting that $\hat{R}_\xi \Phi^{(Il)}_{I_0MI_0} = 0$, we get

$$(\hat{R}^2_\xi + \hat{R}^2_\eta)\,\Phi^{(Il)}_{I_0MI_0} = \hat{\mathbf{R}}^2 \Phi^{(Il)}_{I_0MI_0} = (\hat{I}^2 - I_0^2)\,\Phi^{(Il)}_{I_0MI_0}\,[I(I+1) - I_0^2]\,\Phi^{(Il)}_{I_0MI_0}.$$

Using this equality, substituting (A.7) in (A.6), we get, after multiplying by $\Phi^{(I'l')}_{I_0MI_0}$ and integrating over angles, the system of equations (2.3) for the functions f_{Il}, where

$$V_{Il\,I'l'}(r) = \frac{2\mu}{\hbar^2}\int \Phi^{(Il)*}_{I_0MI_0}\left[V(r, \theta) - \frac{2Ze^2}{r}\right]\Phi^{(I'l')}_{I_0MI_0}\, d\Omega\, d\Theta_i.$$

Let us expand the potential energy of interaction of the α particle with the nucleus in Legendre polynomials:

$$\frac{2\mu}{\hbar^2}\left[V(r, \theta) - \frac{2Ze^2}{r}\right] = \sum_{L=0}^{\infty} V_L(r)\, P_L(\mu), \quad \mu = \cos\theta,$$

and substitute this expression and (A.4) in the formula defining $V_{IlI'l'}(r)$. We then get expression (2.4) for the $V_{IlI'l'}$, where

$$Q_L(Il;\, I'l') = \int \Phi^{(Il)*}_{I_0MI_0} P_L \Phi^{(I'l')}_{I_0MI_0}\, d\Omega\, d\Theta_i = \frac{\sqrt{(2I+1)(2I'+1)}}{2I_0+1} \times \sqrt{\frac{2l'+1}{2l+1}}\, C^{l0}_{L0;\,l'0} \sum_{\Omega} C^{I_0,\,I_0+\Omega}_{l'\Omega;\,I'I_0}\, C^{I_0,\,I_0+\Omega}_{l\Omega;\,II_0}\, C^{l\Omega}_{l'\Omega;\,L0}.$$

Summation over Ω using the well known formula of Racah[7] gives the value (2.4) for Q_L. Substitution of (A.4) in (A.7) gives

$$\psi_{I_0M} = \sum_{Il\Omega} \sqrt{\frac{2I+1}{2I_0+1}} C^{I_0,\, I_0+\Omega}_{l\Omega;\, II_0} \frac{f_{Il}(r)}{r} Y_{l\Omega}(\theta, \varphi)\, \widetilde{D}^{(I_0)}_{M,\, I_0+\Omega}(\Theta_i). \tag{A.8}$$

On the nuclear surface, where $r = R(\theta)$, the wave function should not depend on φ. This will be the case only if, when we substitute $r = R(\theta)$ in (A.8), all terms in the sum over Ω except the term for $\Omega = 0$ vanish.

It then follows from (A.8) that at the nuclear surface

$$\psi_{I_0M}[R(\theta), \theta, \varphi;\ \Theta_i] = \chi(\theta)\, \widetilde{D}^{(I_0)}_{MI_0}(\Theta_i), \tag{A.9}$$

where

$$\chi(\theta) = \sum_{Il} \sqrt{\frac{2I+1}{2I_0+1}} C^{II_0}_{l0;\, II_0} \frac{f_{Il}[R(\theta)]}{R(\theta)} Y_{l0}(\theta)$$

is some function of θ whose specific form will depend on the structure of the α-particle function in the interior of the nucleus. Substituting $r = R(\theta)$ in (A.8), equating the right sides of (A.8) and (A.9), multiplying both sides of the resulting equations by $\{\widetilde{D}^{(I_0)}_{M,I_0+\Omega}(\Theta_i)\}^*$ and integrating over Θ_i, we get the boundary condition (2.5).

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PROPAGATION OF DETONATION WAVES IN THE PRESENCE OF A MAGNETIC FIELD

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It is shown that relativistic detonation waves in a magnetic field possess properties similar to those of the ordinary waves. Solutions of the equations at the discontinuity are presented for the relativistic and nonrelativistic cases.

SHOCK waves in a plasma situated in a magnetic field have been discussed frequently in recent times. In some of this work, for example that of Hoffman and Teller,[1] use was made of the relativistic hydrodynamic equations.

In the present article we consider "perpendicular" detonation waves, i.e., waves propagated at right angles to the direction of the magnetic field. One may expect that the influence of the magnetic field will become noticeable when its energy per unit mass of the medium becomes comparable with the energy liberated in the medium. The calculations are made in a relativistic manner, although for those thermonuclear fuels which are now known