SCATTERING OF SLOW NEUTRONS BY DEUTERONS

V. N. EFIMOV

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

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The neutron-deuteron scattering lengths were computed by a variational method, taking into account deformation of the deuteron. The radial dependence of the nuclear force potential was chosen to be a Gaussian, with parameters adjusted to agree with low energy data on the n-p interaction. The computation was done only for Serber-type exchange forces. The values of the scattering lengths agree with the experimental values within the experimental errors.

AT low energies, the scattering of neutrons by deuterons is characterized by two scattering lengths: a_4 for total spin $S = 3/2$ and a_2 for $S = 1/2$. Analysis of the experimental data gives two possible sets of values of a_4 and a_2:[1]

$$a_4 = 6.2 \cdot 10^{-13}\ \text{cm}; a_2 = 0.8 \cdot 10^{-13}\ \text{cm} \quad (1)$$

or

$$a_4 = 2.4 \cdot 10^{-13}\ \text{cm}; a_2 = 8.3 \cdot 10^{-13}\ \text{cm}. \quad (2)$$

The scattering lengths have been calculated by different methods in various theoretical papers.[2-7] In calculating the s-phases of low energy n-d scattering, Christian and Gammel[5] used the Hulthen[8] variational method and assumed that the deuteron is not distorted by the incoming neutron. The appropriate Euler equations were then solved and values of $a_4 = 5.9 \times 10^{-13}$ cm, $a_2 = 1.5 \times 10^{-13}$ cm were found for the scattering lengths.

Burke and Robertson[7] made use of the approximation of an undistorted deuteron directly in the Schrödinger equation and computed the scattering lengths for several values of the radius of nuclear forces. Their results are in poorer agreement with experiment than those of reference 5. The solutions found in references 5 and 7 require complicated calculations which have to be done on electronic computers, but the values found for the scattering lengths are not in sufficiently good agreement with experiment.

Verde[3] and Clementel[4] used the direct variational method, which leads to less complicated calculations. However, their results are not satisfactory, apparently owing to too crude an approximation to the deuteron wave function and to the assumption that the deuteron is not deformable. Even though Verde's results are in agreement with the experimental values (2) for the scattering lengths, from later work[5-7] and from considerations of symmetry of the wave function which are presented in reference 9, it appears that the set in (1) are the correct values. It is therefore of interest to obtain the correct values of the scattering lengths by a direct variational method taking into account the deformation of the deuteron. Instead of using Hulthèn's method,[8] it seemed to us more effective to start from the variational principle for nucleon-nucleon scattering phases which was given by Rubinow.[10] According to Rubinow, the variation of a certain functional is zero for the solution of the wave equation, and its extremal value determines the scattering phase. The functional depends only on "interior" wave functions which are different from zero only within the range of action of the scattering potential. Such functions are easily approximated by damped exponentials. The generalization of Rubinow's variational method to n-d scattering is presented below.

Let the index 1 refer to the incident neutron, and 2, 3 to the neutron and proton forming the deuteron. $\mathbf{r}_i$ is the radius vector to the i-th nucleon; M is the nucleon mass; E is the kinetic energy of the neutron in the center-of-mass system, E_d the binding energy of the deuteron; S is the total spin of the system, equal to $3/2$ or $1/2$; σ_i are the spin variables, which take on values $\pm 1/2$. The nucleon-nucleon interaction is assumed to be central and charge-invariant:

$$V(ik) = U(ik)(w + bB_{ik} + mM_{ik} + hH_{ik}), \quad (3)$$

where U(ik) is a function of the distance $|\mathbf{r}_i - \mathbf{r}_k|$ between the i-th and k-th nucleons; w, b, m, and h are respectively the fractions of Wigner, Bartlett,

Majorana, and Heisenberg forces, $w + b + m + h = 1$, B_{ik}, M_{ik}, and H_{ik} are the corresponding exchange operators. Introducing the coordinates

$$\mathbf{r} = (\sqrt{3}/2)(\mathbf{r}_3 - \mathbf{r}_2),\ \mathbf{q} = -\mathbf{r}_1 + {}^1/_2(\mathbf{r}_2 + \mathbf{r}_3)$$

the Schrödinger equation takes the form

$$(\Delta_r + \Delta_q + k^2 - k_d^2 + W)\,\Psi^{(S)}(\mathbf{r}, \mathbf{q}, \sigma_1, \sigma_2, \sigma_3) = 0, \quad (4)$$

where

$$k^2 = 4ME/3\hbar^2,\ k_d^2 = 4ME_d/3\hbar^2,$$
$$W = V(12) + V(13) + V(23),$$

Δ_r and Δ_q are the Laplacians with respect to the variables $\mathbf{r}$ and $\mathbf{q}$, respectively.

The functions $\Psi^{(S)}$ are assumed to be everywhere finite, continuous, and antisymmetric under interchange of the neutrons, and to have the following asymptotic form as $q \to \infty$:

$$\Psi^{(3/2)} \approx \chi^{(3/2)}(\sigma_1, \sigma_2, \sigma_3)(f\cot\delta_4 + g), \quad (5a)$$

$$\Psi^{(1/2)} \approx \chi^{(1/2)}(\sigma_1, \sigma_2, \sigma_3)(f\cot\delta_2 + g), \quad (5b)$$

where δ_4 and δ_2 are the s-scattering phases in the states with $S = 3/2$ and $S = 1/2$, respectively;

$$f = \frac{1}{2\pi^{1/2}}\varphi_d(r)\frac{1}{q}\sin kq,\ g = \frac{1}{2\pi^{1/2}}\varphi_d(r)\frac{1}{q}\cos kq, \quad (6)$$

$\varphi_d(r)$ is the wave function of the deuteron,

$$\chi^{(3/2)} = \alpha_1(\sigma_1)\alpha_2(\sigma_2)\alpha_3(\sigma_3),$$

$$\chi^{(1/2)} = \{\alpha_1(\sigma_1)[\alpha_2(\sigma_2)\beta_3(\sigma_3) + \beta_2(\sigma_2)\alpha_3(\sigma_3)]$$
$$- 2\beta_1(\sigma_1)\alpha_2(\sigma_2)\alpha_3(\sigma_3)\}/\sqrt{6}.$$

α and β are defined as usual:

$$\alpha({}^1/_2) = 1,\ \alpha(-{}^1/_2) = 0,\ \beta({}^1/_2) = 0,\ \beta(-{}^1/_2) = 1.$$

We expand the functions $\Psi^{(S)}$ in eigenfunctions of the total spin S and its z projection S_z. These functions have the form:

$$\text{for } S = 3/2,\ S_z = 3/2,\ \chi^{(3/2)} = \alpha_1(\sigma_1)\alpha_2(\sigma_2)\alpha_3(\sigma_3); \quad (7)$$

$$\text{for } S = 1/2,\ S_z = 1/2,$$

$$\chi_1^{(1/2)} = [\beta_1(\sigma_1)\alpha_2(\sigma_2) - \alpha_1(\sigma_1)\beta_2(\sigma_2)]\alpha_3(\sigma_3)/\sqrt{2}; \quad (8a)$$

$$\chi_2^{(1/2)} = \{[\alpha_1(\sigma_1)\beta_2(\sigma_2) + \beta_1(\sigma_1)\alpha_2(\sigma_2)]\alpha_3(\sigma_3) \quad (8b)$$
$$-2\alpha_1(\sigma_1)\alpha_2(\sigma_2)\beta_3(\sigma_3)\}/\sqrt{6},$$

where the function (8a) is antisymmetric and (7) and (8b) are symmetric in the spins of the two neutrons. Then

$$\Psi^{(3/2)} = \chi^{(3/2)}\psi(\mathbf{r}, \mathbf{q}), \quad (9)$$

$$\Psi^{(1/2)} = \chi_1^{(1/2)}\varphi_1(\mathbf{r}, \mathbf{q}) + \chi_2^{(1/2)}\varphi_2(\mathbf{r}, \mathbf{q}). \quad (10)$$

Substituting (9) and (10) in (4) and separating spin variables, we get for $\psi(\mathbf{r}, \mathbf{q})$ the equation

$$(\tau + W_4)\psi = 0, \quad (11)$$

where

$$\tau = \Delta_r + \Delta_q + k^2 - k_d^2,\ W_4 = (\chi^{(3/2)}, W\chi^{(3/2)}),$$

while for $\varphi_1(\mathbf{r}, \mathbf{q})$ and $\varphi_2(\mathbf{r}, \mathbf{q})$ we have the pair of equations

$$(T + W_2)\varphi = 0, \quad (12)$$

where

$$T = \begin{pmatrix}\tau & 0\\ 0 & \tau\end{pmatrix},\qquad W_2 = \begin{pmatrix}W_{11} & W_{12}\\ W_{21} & W_{22}\end{pmatrix},$$
$$\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},\ W_{ik} = (\chi_i^{(1/2)}, W\chi_k^{(1/2)}).$$

To construct total wave functions $\Psi^{(S)}$ which are antisymmetric under interchange of the two neutrons, we need the following combinations of the functions f and g defined in (6):

$$F_4 = (1 - P_{12})f,\ G_4 = (1 - P_{12})g,$$
$$F_2^{(1)} = -(\sqrt{3}/2)(1 + P_{12})f,\ G_2^{(1)} = -(\sqrt{3}/2)(1 + P_{12})g,$$
$$F_2^{(2)} = -{}^1/_2(1 - P_{12})f,\ G_2^{(2)} = -{}^1/_2(1 - P_{12})g, \quad (13)$$

where P_{12} is the exchange operator for the coordinates of the two neutrons.

If in place of ψ, φ_1, φ_2 we introduce new functions Y_4, $Y_2^{(1)}$ and $Y_2^{(2)}$:

$$\psi = F_4\cot\delta_4 + G_4 - Y_4, \quad (14)$$

$$\varphi_1 = F_2^{(1)}\cot\delta_2 + G_2^{(1)} - Y_2^{(1)},\ \varphi_2 = F_2^{(2)}\cot\delta_2 + G_2^{(2)} - Y_2^{(2)}$$

and impose the conditions:

$$P_{12}Y_4 = -Y_4,\ P_{12}Y_2^{(1)} = Y_2^{(1)},\ P_{12}Y_2^{(2)} = -Y_2^{(2)}, \quad (15)$$

we find that all the functions Y and G are finite and continuous, and all the Y's go to zero for $q \to \infty$, so that it follows from (7) to (10), (13), and (14) that $\Psi^{(3/2)}$ and $\Psi^{(1/2)}$ are finite, continuous, have the required symmetry and take on the form (5) as $q \to \infty$.

Substituting (14) in (11) and (12), we get the equations for Y_4, $Y_2^{(1)}$ and $Y_2^{(2)}$:

$$(\tau + W_4)(Y_4 - G_4)$$
$$-\cot\delta_4[W_4F_4 - (1 - P_{12})U(23)f] = 0, \quad (16)$$
$$(T + W_2)(Y_2 - G_2) - \operatorname{ctg}\delta_2(W_2 - U_2)F_2 = 0.$$

Here

$$F_2 = \begin{pmatrix}F_2^{(1)}\\ F_2^{(2)}\end{pmatrix},\ G_2 = \begin{pmatrix}G_2^{(1)}\\ G_2^{(2)}\end{pmatrix},\ Y_2 = \begin{pmatrix}Y_2^{(1)}\\ Y_2^{(2)}\end{pmatrix},$$

$$U_2 = \begin{pmatrix}{}^1/_2(1 + P_{12})U(23) & (\sqrt{3}/2)(1 - P_{12})U(23)\\ (1/2\sqrt{3})(1 - P_{12})U(23) & {}^1/_2(1 + P_{12})U(23)\end{pmatrix},$$

and U(ik) is defined in (3). In deriving (16) we used the fact that F_4 and F_2 satisfy the equations

$$\tau F_4 + (1 - P_{12}) U(23) f = 0, \quad (T + U_2) F_2 = 0, \quad (17)$$

which follow from (6) and (13).

The set of equations (16) and (17) enable us to find the phases δ_4 and δ_2 by using the variational principle[10] which states that the solutions of equation (16) annul the variation of the functional L(Y − G), while $<L>_{extr.} = k \cot \delta$. The functional L has the form:

$$L = k \cot \eta (1 + B)^2 - C, \quad (18)$$

where η is the s-phase shift in Born approximation; the quantities B, C for $S = 3/2$ are given by the formulas:

$$B = \frac{1}{2k} \int d\tau (Y_4 - G_4) \{W_4 F_4 - (1 - P_{12}) U(23) f\},$$

$$C = \frac{1}{2} \int d\tau (Y_4 - G_4) (\tau + W_4) (Y_4 - G_4), \quad (d\tau = d\mathbf{r}\, d\mathbf{q}),$$

and for $S = 1/2$ by

$$B = \frac{1}{2k} \int d\tau [(Y_2 - G_2), (W_2 - U_2) F_2],$$

$$C = \frac{1}{2} \int d\tau [(Y_2 - G_2), (T + W_2)(Y_2 - G_2)],$$

where

$$[A, B] = A_1^* B_1 + A_2^* B_2, \quad \text{if} \quad A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

The computation of the scattering lengths was done with the potential used in reference 5:

$$U(r) = U_0 \exp(-\lambda^2 r^2), \quad U_0 = 86.4 \text{ Mev}, \quad (19)$$

$$\lambda^{-1} = 1.332 \cdot 10^{-13} \text{ cm}$$

and the Serber exchange force. The potential (19) was chosen to agree with the data on the n-p interaction at low energies. For the potential (19), the approximate deuteron function has the form:[5]

$$\varphi_d(x) = 0.02133 e^{-0.03x^2} + 0.08582 e^{-0.16x^2} + 0.18115 e^{-0.76x^2}, \quad (20)$$

where $x = \lambda r$.

The minimum of the functional (18) was found by the direct Ritz method, using trial functions satisfying the conditions (15). For $S = 3/2$, we chose:

(1) in first approximation, neglecting deformation of the deuteron:

$$Y_4 = (1 - P_{12}) \varphi_d(r) \Big[\exp(-\lambda^2 q^2)/2\pi^{1/2} q + \sum_{n=1}^{N} C_n \exp(-n\lambda^2 q^2) \Big], \quad (21)$$

(2) in second approximation, including deformation of the deuteron:

$$Y_4 = (1 - P_{12}) \Phi,$$

where P_{12} is the operator which interchanges the coordinates of the neutrons,

$$\Phi = \varphi_d(r) \exp(-\lambda^2 q^2)/2\pi^{1/2} q + \sum_{p,n} C_{pn} \exp(-\gamma_p \lambda^2 r^2 - n\lambda^2 q^2), \quad (22)$$

$$p = 1, 2, 3; \quad n = 1, 2 \ldots, N;$$

$$\gamma_1 = 0.03, \quad \gamma_2 = 0.16, \quad \gamma_3 = 0.76.$$

For each n, the sum over p in (22) contains the same exponentials $\exp(-\gamma_p x^2)$ as in (20), but the coefficients are varied.

For $S = 1/2$ the trial functions took into account deformation of the deuteron:

(1) In first approximation, the deformation was included as in (22)

$$Y_2^{(1)} = -(\sqrt{3}/2)(1 + P_{12}) \Phi, \quad Y_2^{(2)} = -1/2 (1 - P_{12}) \Phi.$$

(2) In the second approximation, in the spin state (8a) we added an unknown multiple of a function which is symmetric in the coordinates of all the nucleons. The introduction of such a function takes account of the fact that in the state (8a) the neuteron penetrates deep into the deuteron and all three nucleons interact strongly with one another. We chose this function in the form

$$v = \sum_{m=1}^{M} A_m \exp[-m\lambda^2 (r^2 + q^2)/2]. \quad (23)$$

Thus in this case the trial function had the form

$$Y_2^{(1)} = -(\sqrt{3}/2)(1 + P_{12}) \Phi - v,$$

$$Y_2^{(2)} = -1/2 (1 - P_{12}) \Phi.$$

The computational results for the scattering lengths a_4 and a_2 are given in the table, in which the values of M and N indicate the number of terms included in the sums which appear in the trial functions (21), (22), and (23). It is clear that

Dependence of a_4 and a_2 on number of terms in the trial functions (21), (22) and (23). For each n in (22), p runs through the values 1, 2, 3.

	First approximation	Second approximation	Experiment[1]
$a_4 \cdot 10^{13}$ cm^{-1}	$N=1$ 11.8 $N=2$ 7.5 $N=3$ 7.5	$N=1$ 7.7 $N=2$ 6.3 $N=3$ 6.3	6.2±0.2
$a_2 \cdot 10^{13}$ cm^{-1}	$N=1$ 1.7 $N=2$ 1.7	$N=1$, $M=2$ } 1.1 $N=2$, $M=3$ } 1.1	0.8±0.3

the contributions of the terms in the trial functions drop off rapidly with increasing n and m, and that $a_4 = 6.3 \times 10^{-13}$ cm and $a_2 = 1.1 \times 10^{-13}$ cm coincide within the limits of error with the experimental values (1) for the scattering lengths.

We are now in the course of calculation of the energy dependence of the s-phase, the depth of the potential and the type of exchange force, as well as the analysis of low energy proton-deuteron scattering, by the present method.

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19

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NONEQUILIBRIUM PROCESSES IN IMPURITY SEMICONDUCTORS

V. P. SHABANSKII

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

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A treatment is given of the kinetic equations for impurity semiconductors, which describe transitions from the impurity levels to the conduction band. On the assumption that the distribution function of the free electrons (or holes) has the form of an equilibrium distribution function with a certain effective temperature that can be determined from the equations, explicit expressions are given for the energy and kinetic coefficients for cases in which the lifetime of the electrons in the conduction band is determined by photorecombination and triple-collision recombination processes. Nonradiative transitions other than those occurring in triple recombination are included by a phenomenological method. In this case the kinetic and energy coefficients can be expressed in terms of the lifetime of the electrons against such transitions in the equilibrium state. The equations obtained make it possible to determine the electron temperature and the number of electrons in the conduction band in various nonequilibrium processes.

1. THE KINETIC EQUATIONS FOR THE FREE ELECTRONS, INCLUDING EFFECTS OF RECOMBINATION AND IONIZATION

THE kinetic equations for the distribution function $\bar{n}$ of electrons or holes, including effects of their possible heating up, have the form [cf. e.g., Eqs. (3.1) ff. in reference 1]

$$\frac{\partial n_0}{\partial t} + \frac{v}{3}\nabla\hat{\mathbf{n}}_1 + \frac{1}{N_e}\frac{\partial}{\partial\varepsilon}(N_e S_1) + \Sigma_{ee}(\bar{n}_0, \bar{n}_0) + \varphi(\varepsilon) = 0, \tag{1.1}$$

$$\frac{\partial\bar{\mathbf{n}}_1}{\partial t} + v\nabla\bar{n}_0 + e\mathbf{E}v\frac{\partial\bar{n}_0}{\partial\varepsilon} + \frac{e}{mc}[\mathbf{H}\times\bar{\mathbf{n}}_1] + \frac{v}{L}\bar{\mathbf{n}}_1 = 0. \tag{1.2}$$