REFLECTION FROM A BARRIER IN THE QUASI-CLASSICAL APPROXIMATION. II

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An asymptotic expression for the coefficient of reflection from a barrier, valid for the whole range of the parameters U_0/E and k_0a, has been obtained using the condition for the quasi-classical approximation $k_0a \gg 1$ and the condition $k_0a(E - U_0)/U_0 \gg 1$.

It was shown in a previous paper[1] that the amplitude of reflection R from a one-dimensional potential barrier has the following form in the quasi-classical approximation:

$$R = -i\exp\left\{\frac{2i}{\alpha}\int^{\xi_0} k d\xi\right\}. \tag{1}$$

We follow the notation of reference 1. It was shown that formula (1) is applicable in that region of the parameters $\kappa = U_0/E$ and $\alpha = 1/k_0a$ where perturbation theory is inapplicable. For example, if the potential U has only poles of first order, formula (1) is valid for $\kappa/\alpha \gg 1$, whereas perturbation theory is applicable for $\kappa/\alpha \ll 1$. Here the intermediate region $\kappa/\alpha \sim 1$ is not included in the investigation. The aim of the present paper is to present an asymptotic formula for the reflection amplitude which is valid in the whole range of the parameters κ, α, $(1-\kappa \gg \alpha)$. The form of this formula depends essentially on the type of singularity of the potential.

We restrict our discussion to the two most important cases: poles of first and of second order.

As in reference 1, we seek the reflection amplitude in the form of the expansion

$$R = -\frac{1}{2i}\left\{V_{-1,1} + \frac{1}{2\pi}\int\frac{V_{-1,k}V_{k,1}}{1-k^2}dk + \frac{1}{(2\pi)^2}\right.$$
$$\left.\times\iint\frac{V_{-1,k_1}V_{k_1,k_2}V_{k_2,1}}{(1-k_1^2)(1-k_2^2)}dk_1\,dk_2\ldots\right\}, \tag{2}$$
$$V_{k,k'} = -\alpha\int_{-\infty}^{\infty}\exp\left\{\frac{i}{\alpha}(k'-k)\tau\right\}q(\tau)\,d\tau.$$

We write the function q in the form

$$q = 5[(k^2)']^2/16k^6 - (k^2)''/4k^4. \tag{3}$$

For small κ the root ξ_0 of the function $k^2 = 1 - U/E$ lies close to the singularity ξ_1 of the potential $U(\xi)$. Let this singularity be a pole of first order. Close to ξ_1 we have then

$$\frac{U}{E} = \kappa\frac{A}{\xi-\xi_1}; \quad k^2 = \frac{\xi-\xi_0}{\xi-\xi_1}, \tag{4}$$

where $\xi_0 = \xi_1 + A\kappa$ and the quantity A is of order unity. By formula (3) we obtain the approximate expression

$$q = \frac{\mu}{16}\frac{8(\xi-\xi_0)+5\mu}{(\xi-\xi_1)(\xi-\xi_0)^3}, \quad \mu = A\kappa. \tag{5}$$

For $\tau = \int^{\xi} k\,d\xi$ we get

$$\frac{\tau-\tau_0}{\mu} = \sqrt{\frac{(\xi-\xi_0)}{\mu}\frac{(\xi-\xi_1)}{\mu}}$$
$$-\frac{1}{4}\ln\left(\sqrt{\frac{\xi-\xi_0}{\mu}} + \sqrt{\frac{\xi-\xi_1}{\mu}}\right). \tag{6}$$

With the notations

$$(\tau-\tau_0)/\mu = t, \quad (\xi-\xi_0)/\mu = \eta,$$

we obtain:

$$t = f(\eta), \tag{7}$$

$$f(\eta) = \sqrt{\eta(1+\eta)} - \frac{1}{4}\ln(\sqrt{\eta} + \sqrt{1+\eta}). \tag{8}$$

We denote by g the inverse function of f. We have

$$\eta = g(t); \quad q = (8g(t)+5)/16\mu^2[1+g(t)]g^3(t). \tag{9}$$

Computing the leading term of the matrix element, we obtain

$$V_{k,k'} = -\frac{\alpha}{16\mu}\exp\left\{\frac{i\rho(k'-k)}{\alpha} - \frac{\sigma|k'-k|}{\alpha}\right\}$$
$$\times\int_C\frac{8g(t)+5}{[1+g(t)]g^3(t)}\exp\left\{\frac{i(k'-k)\mu}{\alpha}t\right\}dt \tag{10}$$
$$(\tau_0 = \rho + i\sigma),$$

where the integral is taken along a contour C which encloses the singular points of the expression under the integral sign. We use the notation

$$F_1(x) = -\frac{1}{16x}\int\limits_C \frac{8g(t)+5}{[1+g(t)]\,g^3(t)}\, e^{ixt}\,dt. \tag{11}$$

Then formula (10) takes the form:

$$V_{k,k'} = (k'-k)\,F_1\left((k'-k)\frac{\mu}{\alpha}\right) \times \exp\left\{\frac{i\rho\,(k'-k)}{\alpha} - \frac{\sigma\,|k'-k|}{\alpha}\right\}. \tag{10'}$$

In computing the reflection amplitude (2) we restrict the integration to the interior of the region $-1 \le k_1 \le k_2 \le \ldots \le k_n \le 1$. The function $F_1(x)$ is small x proportional to x. Therefore all integrals converge, and the sum of the series (2) is represented in the form

$$R = F\left(\frac{\mu}{\alpha}\right)\exp\left\{\frac{2i}{\alpha}\int^{\xi_0} k\,d\xi\right\}, \tag{12}$$

where $F(x)$ is a universal function independent of the form of the potential $U(\xi)$. We find the function $F(x)$ with the help of the known exact solution of the Schroedinger equation with the potential $U = U_0/(1+e^{-\xi})$ (reference 2):

$$F(x) = -\frac{2\pi i \exp\{ix\ln(x/2e) - \pi x/2\}}{\Gamma(ix/2)\,\Gamma(1+ix/2)}. \tag{13}$$

The absolute value of this function is equal to

$$|F(x)| = 2\sinh\frac{\pi x}{2}\, e^{-\pi x/2}. \tag{14}$$

It is easily seen that for small values $x = A\kappa/\alpha$, the function $|F(x)| \approx \pi x$, and for large x, $|F(x)| \approx 1$.

In the case of a pole of second order, analogous considerations lead to the formula

$$R = G\left(\frac{\sqrt{\mu}}{\alpha}\right)\exp\left\{\frac{2i}{\alpha}\int^{\xi_0} k\,d\xi\right\}, \tag{15}$$

where $G(x)$ is the universal function

$$|G(x)| = 2\cosh\left(\pi\sqrt{x^2-\frac{1}{4}}\right)e^{-\pi x}. \tag{16}$$

(Here we made use of the known exact solution of the Schroedinger equation with the potential $U = U_0/\cosh^2\xi$ (reference 2).)

Schiff and Saxon[3,4] investigated the potential scattering of particles with high energies in three dimensions under the assumption $\kappa/\alpha \sim 1$, $\alpha \ll 1$. The results of the present paper show that the methods of these authors cannot be applied to the scattering into large angles, when the cross sections are exponentially small. In the one-dimensional case it can indeed be easily shown that their method leads to the following expression for the reflection amplitude R (the potential is assumed to have a pole of first order):

$$|R| = \frac{\pi\mu}{\alpha}\left|\exp\left\{\frac{2i}{\alpha}\int^{\xi_0} k\,d\xi\right\}\right|.$$

Comparing this expression with (12) and (14), we see that the results agree only for small κ/α, i.e., in the region where perturbation theory is applicable. The error arises from the fact that in their calculations Schiff and Saxon neglect integrals along the real axis over small quantities of order κ^2/α and higher multiplied by strongly oscillating functions. However, these integrals give a contribution comparable to the included terms.

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