In the table a comparison is given between the theoretical values of the excitation energies of the first and second rotational-vibrational bands of excited states of odd nuclei and experimental data. There the values of the parameters $\hbar\omega_0$ and δ, used in the calculation of theoretical values, are also given.

Comparing the spectrum of collective excitations of odd nuclei with the spectrum of collective excitation of even-even nuclei, it is possible to draw the following conclusions: (1) The break-up of collective excitations into a system of rotational-vibrational bands in odd nuclei sets in for lower values of δ than in even-even nuclei; (2) The values of the parameter ω_0, which can be called the frequency of vibration of the nuclear surface, in the ground state is smaller in odd nuclei than in even-even nuclei having the same value of the parameter δ.

For $\delta > 3$ the quantity ν takes on values near to integral ones 0, 1, 2, ...; further, according to Eq. (1.11a) one can approximately set

$$\xi = 1 + g\,[J(J+1) - K(K+1)]/3\delta^4.$$

Then Eq. (1.18) can be replaced by the approximate equality

$$\varepsilon/\hbar\omega_0 = (\nu + 1/2) + [J(J+1) - K(K+1)]/6\delta^2 - a\,[J(J+1) - K(K+1)]^2/\delta^6. \quad (2.1)$$

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BEHAVIOR OF THE DISTRIBUTION FUNCTION OF A MANY-PARTICLE SYSTEM NEAR THE FERMI SURFACE

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The form of the distribution function of a system of electrons is studied in the Hartree approximation near the Fermi surface, for the case of a weakly inhomogeneous distribution. It is shown that in this region the inhomogeneity has a particularly strong effect, so that the correct expression for the distribution function, as given in this paper, is decidedly different in this region from the expression usually employed (that calculated from the Thomas-Fermi model). It is pointed out that the latter expression is completely unsuitable for use in problems in which the neighborhood of the Fermi surface plays an important part.

As is well known, the distribution function (the density matrix in a mixed representation) is the most important quantity characterizing a many-particle system. By means of it one can calculate without difficulty quite a number of physical quantities for the system in question.

In the present note we study in the Hartree approximation the form of the distribution function in the region of phase space near the Fermi surface. Our specific problem is that of a system of nonrelativistic electrons in a stationary state at temperature zero, and we confine ourselves to the case in which the occupation numbers of the levels depend only on the energy.

If our system is characterized by a sufficiently smooth distribution of the density and potential energy, namely if the condition for quasi-classical behavior,

$$\xi \equiv |\nabla(p_0^2)|/p_0^3 \ll 1, \tag{1}$$

is satisfied, then it is common practice to use for the distribution function the following expression, which corresponds to the Thomas-Fermi model:*

$$f(\mathbf{r}, \mathbf{p}) = 2(2\pi)^{-3}\theta(p^2 - p_0^2(\mathbf{r})), \tag{2}$$

$$\theta(x) = {}^1/_2(1 - x/|x|).$$

Here p_0 is the Fermi limiting momentum, related to the sum $\Phi(\mathbf{r})$ of the potentials of the external and self-consistent fields and the limiting energy E_0 by the equation $p_0^2(\mathbf{r}) = 2(E_0 - \Phi(\mathbf{r}))$. We use throughout atomic units with $e = \hbar = M = 1$.

It will be shown below (cf. also reference 1) that near the Fermi surface, namely for

$$|p - p_0| \sim \sqrt{\xi}\, p_0, \tag{3}$$

the expression (2) is not a useful one even when Eq. (1) holds, since in this region effects of the inhomogeneity of p_0^2 become very pronounced. It turns out to be possible to find an expression for f which is acceptable also in the region (3).

It is convenient to start from the operator expression for f in the Hartree approximation[1,2]

$$f(\mathbf{r}, \mathbf{p}) = (2\pi)^{-3} 2 \langle \theta(\hat{p}^2 - p_0^2(\mathbf{r}))\rangle_{\mathbf{p}}, \tag{4}$$

where for an arbitrary operator $\hat{a}$

$$\langle \hat{a}\rangle_{\mathbf{p}} \equiv \exp(-i\mathbf{pr})\,\hat{a}\exp(i\,\mathbf{pr}).$$

Apart from an unimportant factor the argument of the function θ is equal to the quantity $\hat{H} - E_0$, where $\hat{H} = \hat{p}^2/2 + \Phi(\mathbf{r})$ is the Hamiltonian. Expression (4) thus corresponds to a step-function distribution in the energy space; it is clear, however, that in the phase space this distribution must inevitably be smeared out because of the fact that the coordinates and momentum do not commute with the Hamiltonian. This fact is displayed formally in the failure to commute of the operators $\hat{p}$ and p_0^2 in the argument of the θ function in Eq. (4). If we denote by $\hat{K}_n$ the corresponding n-th order commutator, then we have for small ξ, in order of magnitude*

$$\langle \hat{K}_n\rangle_{\mathbf{p}} \sim p_0^{2n+2}\xi^n.$$

On the other hand, the quasi-classical value of the argument of the θ function, corresponding to complete neglect of the commutators, is given by $p^2 - p_0^2$. From this it follows that the ratios

$$\varkappa_n \equiv \langle \hat{K}_n\rangle_{\mathbf{p}}/|p^2 - p_0^2|^{n+1}. \tag{5}$$

are dimensionless parameters that determine the part played by the commutators of various orders, and thus also determine the importance of the inhomogeneity.

If the phase point lies far enough from the Fermi surface so that $|p^2 - p_0^2| \sim p_0^2$, then for small ξ we have $\kappa_n \sim \xi^n$, and we actually arrive at Eq. (2). Near the Fermi surface, however, κ_n increases because of the decrease of the denominator, and the inhomogeneities begin to play a decisive role.

To find the region of phase space in which the commutators must not be neglected, it suffices to equate κ_n to a quantity of the order unity. Then the difference $p^2 - p_0^2$ defining the size of this region will be equal to the largest of the quantities

$$|\langle \hat{K}_n\rangle_{\mathbf{p}}|^{1/(n+1)} \sim p_0^2\xi^{n/(n+1)}, \quad n = 1 \div \infty.$$

For small ξ we must take $n = 1$, which at once leads to the estimate (3).

The next question is to find out the relative importance of commutators of different orders. The answer is provided by the values of the ratios

$$\varkappa_n/\varkappa_1 \sim [p_0^2\xi/|p^2 - p_0^2|]^{n-1}. \tag{6}$$

In the region (3) in which we are interested these ratios are of the order $\xi^{(n-1)/2}$ and vanish for $\xi \to 0$, which shows that only the first commutator needs to be taken into account. The region in which other commutators also become important is considerably narrower:

$$|p - p_0| \sim \xi p_0. \tag{7}$$

The contribution of this region is usually small because of its small width.

It must be noted that the behavior of the distribution function just at the Fermi surface has been studied by Migdal.[3] There, moreover, the interactions between the particles were taken into account exactly, but only an isolated system was considered, without any external field.

*The factor 2 in this formula corresponds to the two orientations of the spin.

*A characteristic value of the momentum is a quantity of the order of p_0.

Proceeding to a quantitative study of the situation in the region (3), we note that the function of noncommuting arguments appearing in Eq. (4),

$$\theta(\hat{p}^2 - p_0^2) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{t + i\varepsilon} \exp[it(\hat{p}^2 - p_0^2)] \quad (8)$$

can be given a meaning by means of the relations[1,2]

$$\langle \exp[it(\hat{p}^2 - p_0^2)] \rangle_{\mathbf{p}} \quad (9)$$
$$= \exp[it(p^2 - p_0^2)] \langle F(t^2\hat{K}_1, \ldots t^{n+1}\hat{K}_n \ldots) \rangle_{\mathbf{p}},$$

where F is a certain function of the commutators of the operators $t\hat{p}^2$ and tp_0^2, which reduces to unity when the latter commute. According to the estimates given above, which are immediately confirmed if one makes the replacement $t \to t|p^2 - p_0^2|^{-1}$ in Eqs. (8) and (9), in the region (3) we can neglect all the commutators except the first. This makes it possible to use a well known formula of Glauber[4] which takes account of the first commutator and all its powers*

$$\langle F \rangle_{\mathbf{p}} = \langle \exp(t^2\hat{K}_1/2) \rangle_{\mathbf{p}} \approx \exp\{-it^2 \mathbf{p} \nabla (p_0^2)\}. \quad (10)$$

Here we neglect quantities of higher order in ξ. Substituting Eqs. (8) to (10) into Eq. (4), we get

$$f(\mathbf{r}, \mathbf{p}) = (2\pi)^{-3}\{1 - (1 - is)(C(x) + isS(x))\},$$
$$s \equiv \mathbf{p}\nabla(\mathbf{p}_0^2)/|\mathbf{p}\nabla(p_0^2)|, \quad x \equiv (p^2 - p_0^2)/2\,|\mathbf{p}\nabla(p_0^2)|^{1/2}. \quad (11)$$

C and S are the Fresnel integrals.

For reference we present the general formula, which may be useful in the nondegenerate case:

$$\langle \varphi(\hat{p}^2 - p_0^2) \rangle_{\mathbf{p}} = \frac{1 - is}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist^2} \varphi[(p^2 - p_0^2)(1 + t/x)]\,dt, \quad (12)$$

where φ is an arbitrary function.

We also give the expression (11) in a form averaged over the directions of **p**

$$f(\mathbf{r}, \mathbf{p}) = (2\pi)^{-3}\, 2\left[\theta(p^2 - p_0^2)\chi + \frac{1-\chi}{2}\right],$$
$$\chi = (1 + 2y^2)S(y) + (1 - 2y^2)C(y) \quad (13)$$
$$+ \sqrt{\frac{2}{\pi}}\, y(\cos y^2 + \sin y^2),$$
$$y \equiv |p^2 - p_0^2|/2(p|\nabla(p_0^2)|)^{1/2}.$$

For $y \gg 1$ we get the required value $\chi = 1$, but for $y \sim 1$, i.e., in the region 3, the expressions (11) and (13) are decidedly different from Eq. (2). In particular, for small y we have $\chi = 2(2/\pi)^{1/2}y$. The distribution obtained is qualitatively similar to the curve corresponding to the nondegenerate case with the effective temperature

$$kT_{\mathrm{eff}} \sim \sqrt{\xi}\, p_0^2.$$

From this it follows in particular that in the nondegenerate case the effect in question is unimportant for $T \gg T_{\mathrm{eff}}$. We emphasize once again that Eqs. (11) to (13) apply only in the region in which $|p - p_0|/\xi \to \infty$ for $\xi \to 0$.

From what has been said above it follows that if a quantity with which we are concerned and which relates to a system of particles is expressed in terms of an integral of the function f over a sufficiently wide region of the phase space, $|p - p_0| \sim p_0$, then it is permissible to use Eq. (2) if the condition (1) holds.

In a number of problems of many-body theory, however, the decisive part is played by the neighborhood of the Fermi surface, with the width of this region determined by the distribution smeared out by the inhomogeneity, as discussed above. In this region, where $|p - p_0| \sim \xi^{1/2}p_0$, it is necessary to use Eqs. (11) to (13). This is the case, for example, in the problem of calculating exchange and correlation effects, which will be dealt with in special papers.

Finally, if for any reason the decisive role is played by a still narrower neighborhood of the Fermi surface, with $|p - p_0| \sim \xi p_0$, the problem becomes much more complicated, and it is evidently impossible to give a closed expression for f in this case.

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*At points where the first commutator vanishes, the main role will be played by commutators of higher order. Nevertheless this fact is unimportant, since ordinarily we are interested in integral expressions, to which these points contribute very little.

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