ENERGY SPECTRUM OF A HIGH DENSITY ELECTRON GAS

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The procedure of approximate second quantization is used to sum the special class of diagrams corresponding to the divergent terms in the standard perturbation theory expansion for a system with coulomb interaction (infrared divergence). An energy spectrum for a high density electron gas is derived, which contains both individual excitations (quasi-particles obeying Fermi statistics) and collective excitations of the plasma type. The correction to the specific heat due to the self energy of the elementary excitations is computed. The results are compared with those of Gell-Mann and Brueckner. To a certain extent, the results can be considered to be a justification of the independent particle model in the electron theory of metals.

1. INTRODUCTION

SOME of the ideas of quantum field theory, in particular the method of graphical analysis of matrix elements, prove to be very fruitful in quantum statistics. In papers of Gell-Mann and Brueckner[1] and Sawada,[2] for the case of a high density electron gas, a partial summation is carried out over the specially selected diagrams corresponding to the most strongly divergent terms in the perturbation expansion, thus eliminating the well known "infrared catastrophe" which is related to the long range character of the coulomb interaction. Hugenholtz[3] has developed in detail a technique for constructing diagrams for a system of interacting particles obeying Fermi statistics, and has investigated the dependence of various matrix elements on the volume of the system.

In the present paper, we apply the method of approximate second quantization to a high density electron gas, and obtain the energy spectrum of the elementary excitations corresponding to both collective motion of the system and to individual excitations.

In reference 1 it was shown that for a system of Fermi particles at high density (large Fermi momentum), only those matrix elements of the Coulomb interaction are important which correspond to transitions near the Fermi surface. The electrons and holes then form bound pairs. Thus the Gell-Mann — Brueckner partial summation of diagrams in which electron and hole lines form irreducible complexes is equivalent to treating a model dynamical system with a Hamiltonian which is obtained from the exact Hamiltonian by dropping all terms which do not correspond to electron-hole pairs. It was also shown in reference 1 that the matrix elements corresponding to the exchange of an excited pair give a contribution of higher order, and can therefore be dropped. In the present paper the authors start from the results of a paper* of Bogoliubov, Tiablikov, and Tolmachev, in which the idea of partial summation of diagrams is combined with the well developed technique of computation in the method of approximate second quantization. These ideas are proving to be very fruitful and, as is well known, are very sucessful in the theory of superconductivity.[4] Thanks to the convenient form of writing diagonal matrix elements, we obtain in the present paper, in very much simpler fashion, not only the results of Gell-Mann and Brueckner,[1,5] but also explicit expressions for the energy of elementary excitations, which they were unable to get.

2. GENERAL FORMALISM

Let us proceed to formulate the problem mathematically and obtain the general relations from which it is easy to find the spectrum of elementary excitations in which we are interested. We start from the exact Hamiltonian of a system of N particles (electrons) contained in volume Ω:

*This work was reported at the seminar of the Theoretical Physics Division of the Steklov Mathematical Institute of the Academy of Sciences, USSR, in September, 1957.

$$H = \sum_{p\sigma} \varepsilon(p)\, a^*_{p\sigma} a_{p\sigma} + \sum_q \frac{\nu(q)}{2\Omega} \sum_{\substack{pp' \\ \sigma\sigma'}} a^*_{p+q\sigma}\, a^*_{p'-q\sigma'}\, a_{p'\sigma'} a_{p\sigma}; \qquad (1)$$

$a^*_{p\sigma}$ and $a_{p\sigma}$ are the creation and annihilation operators for an electron with momentum p and spin σ. We shall omit the spin indices from now on. As usual, we assume the presence of a uniformly distributed positive charge, which is necessary to maintain the neutrality and equilibrium of the system. All energies are measured in Rydbergs, and momenta in units of the Fermi momentum. Thus the expressions for the kinetic energy of the electron and the Fourier transform of the Coulomb interaction are:

$$\begin{aligned} \varepsilon(p) &= (p^2 p_F^2/2m)/(e^4 m/2\hbar^2) = p^2/(r_s\alpha)^2, \\ \nu(q) &= (4\pi\hbar^2 e^2/q^2 p_F^2)/(e^4 m/2\hbar^2) = 8\pi/k_F^3 r_s \alpha q^2, \end{aligned} \qquad (2)$$

where $r_s = r_0/a$ is a dimensionless parameter, m and e are the mass and charge of the electron, $a = \hbar^2/me^2$, $4\pi r_0^3/3 = \Omega/N$, $p_F = \hbar k_F$ is the Fermi momentum, and $\alpha = (4/9\pi)^{1/2}$. Since r_s is proportional to e^2, the expansion in powers of r_s is the usual perturbation series; r_s will be small if the density is sufficiently high.

If we choose as the "vacuum," with respect to which we define the electron and hole, the state Φ_G in which all individual states within a certain region (G) in momentum space are filled with electrons while all other states (H) are empty, the creation and annihilation operators for the electron and hole will be, respectively:

$$\begin{aligned} c^*_p &= (1 - \varkappa(p))\, a^*_p, \quad b^*_p = \varkappa(p)\, a_p, \\ c_p &= (1 - \varkappa(p))\, a_p, \quad b_p = \varkappa(p)\, a^*_p. \end{aligned} \qquad \varkappa(p) = \begin{cases} 1 & \text{for } p \in G \\ 0 & \text{for } p \in H. \end{cases}$$

Then

$$a_p = c_p + b^*_p, \quad a^*_p = c^*_p + b_p. \qquad (3)$$

Using the ideas in the cited papers of Bogoliubov, Tiablikov, and Tolmachev, we show that the operation of summation of contributions from diagrams of a selected class, corresponding to the most strongly divergent terms in each order of the infinite perturbation series,[1,5] can be replaced by the simpler and well developed method of approximate second quantization. In fact, if we retain in the exact Hamiltonian only terms that can be uniquely associated with fundamental elements (irreducible complexes and vertex parts) of the diagrams of this particular class, it is clear that the exact solution of the problem for the model dynamical system with this simplified Hamiltonian is completely equivalent to the operation of summation of the infinite perturbation series which we described above. In our case, the irreducible complexes in the diagrams to be summed are the electron-hole pair lines, while the vertex parts correspond to scattering of an electron by the hole of a given pair. It is not hard to see that such irreducible complexes correspond to combinations of the type $c^*_{p+q} b^*_p$ and $b_p c_{p+q}$ in the interaction Hamiltonian, and that just these terms should be kept to get the model Hamiltonian.

It should be mentioned that a similar idea was essentially already used by Sawada,[2] but since he did not write the expression for the self energy of the pair explicitly, he did not succeed in representing the complete simplified Hamiltonian as a quadratic form to which the diagonalization method could be applied; to solve the problem he had to use more complicated mathematical methods. In doing this, he omitted the contribution from excitations of the plasma type, so that his results are not entirely correct. These deficiencies are eliminated in the present paper, and new results are obtained.

Substituting (3) in the Hamiltonian (1) and keeping only terms corresponding to irreducible electron-hole complexes, and also dropping exchange terms (an important point here is that we express the diagonal part in terms of the energy $\omega(p, q)$ of a free pair), we get the model Hamiltonian in the form

$$\begin{aligned} \tilde{H} = E(\Phi_G) &- \sum_{pq} \varkappa(p)(1 - \varkappa(p+q)) \frac{\nu(q)}{2\Omega} + \sum_{pq} \omega(p, q)\, \eta^{q*}_p \eta^q_p \\ &+ \sum_q \frac{\nu(q)}{2\Omega} \sum_{pp'} [\eta^q_p \eta^{-q}_{p'} + \eta^{q*}_p \eta^{q*}_{p'} + 2\eta^{q*}_p \eta^q_{p'}], \\ &\omega(p, q) = \varepsilon(p+q) - \varepsilon(p), \end{aligned} \qquad (4)$$

where $\eta^{q*}_p = c^*_{p+q} b^*_p$, $\eta^q_p = b_p c_{p+q}$ are the creation and annihilation operators for an electron-hole pair, defined with respect to the state Φ_G; they satisfy complicated commutation relations, which are different from those for both fermions and bosons. The basic idea of approximate second quantization just consists of the fact that when we treat the state Φ_G and there no particles outside the region G, these operators can be regarded as Bose operators, i.e., when applied to the state Φ_G they satisfy the same commutation relations as the usual Bose amplitudes.

The Hamiltonian (4), which is a quadratic form in η^{q*}_p, η^q_p, can be diagonalized using the familiar method of Tiablikov.[6] To do this, we make a canonical transformation to new Fermi amplitudes

$$\xi_\alpha = \sum_{pq} [u_p^{q*}(\alpha)\,\eta_q^q - v_p^{q*}(\alpha)\,\eta_q^{q*}],$$

$$\sum_{pq} \varkappa(p)(1-\varkappa(p+q))\,[\,|u_\rho^q|^2 - |v_\rho^q|^2] = 1, \qquad (5)$$

$$\xi_\alpha^* = \sum_{pq} [u_\rho^q(\alpha)\,\eta_p^{q*} - v_\rho^q(\alpha)\,\eta_p^q]; \qquad [\xi_\alpha,\ \xi_{\alpha'}^*] = \delta_{\alpha\alpha'}.$$

If u and v satisfy the equations

$$\begin{aligned} E_\alpha u_\rho^q(\alpha) &= \omega(p,q)\,u_\rho^q(\alpha) \\ &+ \frac{\nu(q)}{\Omega}\sum_{p'}\varkappa(p')\,[(1-\varkappa(p'+q))\,u_{\rho'}^q(\alpha) \\ &+ (1-\varkappa(p'-q))\,v_{p'}^{-q}(\alpha)], \\ -E_\alpha v_p^q(\alpha) &= \omega(p,q)\,v_\rho^q(\alpha) \\ &+ \frac{\nu(q)}{\Omega}\sum_{p'}\varkappa(p')\,[(1-\varkappa(p'-q))\,u_{p'}^{-q}(\alpha) \\ &+ (1-\varkappa(p'+q))\,v_{p'}^q(\alpha)], \end{aligned} \qquad (6)$$

the Hamiltonian, when expressed in terms of ξ_α^* and ξ_α takes on the diagonal form

$$\begin{aligned} \tilde{H} = E(\Phi_G) &- \sum_{\alpha,p,q} E_\alpha \left| v_p^q(\alpha) \right|^2 \varkappa(p)(1-\varkappa(p+q)) \\ &- \sum_{pq}\varkappa(p)(1-\varkappa(p+q))\frac{\nu(q)}{2\Omega} + \sum_\alpha E_\alpha \xi_\alpha^*\xi_\alpha. \end{aligned} \qquad (7)$$

Here the E_α are the energies of the elementary excitations, which are given by the zeros of the function

$$\Psi(z,q) = 1 - \frac{\nu(q)}{\Omega}\sum_{p'}\varkappa(p')(1-\varkappa(p'+q))\frac{2\omega(p',q)}{z^2-\omega^2(p',q)}. \qquad (8)$$

The part of the Hamiltonian $\tilde{H}$ which does not contain operators [without $E(\Phi_G)$]

$$\begin{aligned} \Delta E(\Phi_G) = &-\sum_\alpha E_\alpha \sum_{p,q}\frac{\varkappa(p)(1-\varkappa(p+q))}{(E_\alpha+\omega(p,q))^2} \\ &\times\left\{\sum_{p'}\left[\frac{\varkappa(p')(1-\varkappa(p'+q))}{(E_\alpha-\omega(p',q))^2}\right.\right. \\ &\left.\left.-\frac{\varkappa(p')(1-\varkappa(p'+q))}{(E_\alpha+\omega(p',q))^2}\right]\right\}^{-1} - \sum_{pq}\varkappa(p)(1-\varkappa(p+q))\frac{\nu(q)}{2\Omega} \end{aligned} \qquad (9)$$

determines the change in energy of the ground state due to dynamical correlation between electrons.*

3. GROUND STATE ENERGY

The true lowest state corresponds to a completely filled Fermi sphere (Fermi vacuum Φ_V). Substituting in (9)

$$\varkappa(p) = \theta(p),\ \theta(p) = \begin{cases} 1, & \text{for } |p|<1, \\ 0, & \text{for } |p|>1, \end{cases} \qquad (10)$$

and using (8) and the residue theorem, we easily get the expression for the correlation energy in the form*

$$\begin{aligned} \Delta E(\Phi_V) = &-\sum_q \frac{1}{4\pi i} \\ &\times\int_\Gamma\left\{\ln\left[1-\frac{\nu(q)}{\Omega}\sum_p\theta(p)(1-\theta(p+q))\frac{2\omega(p,q)}{z^2-\omega^2(p,q)}\right]\right. \\ &\left.+\frac{\nu(q)}{\Omega}\sum_p\theta(p)(1-\theta(p+q))\frac{2\omega(p,q)}{z^2-\omega^2(p,q)}\right\}dz. \end{aligned} \qquad (11)$$

The contour Γ runs clockwise around the positive real axis. If we deform Γ to run along the imaginary axis from $-\infty$ to $+\infty$, Eq. (11) coincides exactly with the result obtained in reference 1 by partial summation of the divergent perturbation series.

As is well known, the branch of collective excitations of the density-fluctuation type (plasma waves) is very important in the spectrum of a dense charged gas. As already mentioned, excitations of this type contribute to the correlation energy. From the physical point of view, the plasma oscillations are important because they show the effect of the long-range correlation between electrons, which results in the screening of the Coulomb interaction and the cutting off of integrals over virtual momenta at a lowest momentum k_c which is proportional to $r_s^{1/2}$. This is why the idea of summation of divergent diagrams is completely justified mathematically, and is the essential physical reason for the elimination of the infrared divergence, which is the basis of all the investigations discussed in the present paper. Let us show that in our formalism the plasma oscillations are included in the energy spectrum of the system. For $\Omega \to \infty$, we have the asymptotic expression:

$$\begin{aligned} \Psi(z;q) = 1 + \frac{2r_s\alpha}{\pi q^2}\left[1-\frac{z'}{4q}\ln\frac{(z'+2q)^2-q^4}{(z'-2q)^2-q^4}\right. \\ \left.+\frac{1}{2}\left(\frac{1}{q}-\frac{q}{4}-\frac{z'^2}{4q^3}\right)\ln\frac{z'^2-(2q+q^2)^2}{z'^2-(2q-q^2)^2}\right], \qquad z' = (r_s\alpha)^2 z. \end{aligned} \qquad (12)$$

The isolated zero of the function $\Psi(z,q)$ determines the frequency of collective ordered oscillation of the system. For $q \ll 1$ it reduces to the familiar dispersion relation for plasma waves:

$$\omega^2(q) \equiv z^2 \approx \left(\frac{4\pi e^2 n}{m} + \frac{3}{5}\frac{p_F^2}{m^2}q^2 + \cdots\right)\Big/\left(\frac{e^4 m}{2\hbar^2}\right)^2,\ n = \frac{N}{\Omega}. \qquad (13)$$

*The second-order exchange correction is not included in expression (9) for the correlation energy.

*This formula for the case of the Fermi vacuum was obtained in the paper of Bogoliubov, Tiablikov, and Tolmachev which we cited earlier.

Aside from the plasma solution, we find from (8) only the trivial result $E_\alpha = \omega(\mathbf{p}, \mathbf{q})$ for the spectrum of elementary excitations. Thus, in our approximation the excitation energy of a pair is not changed by the interaction, or, in the language of quantum field theory, the excitation has no self energy. This is entirely natural since the matrix elements of the interaction potential corresponding to diagrams with external lines, which are the ones which would give rise to a self energy of the elementary excitations, were dropped in getting the model Hamiltonian.

We also note that η^* and η are not Bose amplitudes with respect to the excited state, since the occupation number of the excited state cannot at all be assumed to be zero. Consequently our model Hamiltonian is not suitable for investigating the spectrum of elementary excitations if we use the Fermi vacuum as the initial state.

4. ENERGY OF EXCITED STATES

This difficulty can be overcome as follows. We shall consider the model Hamiltonian (7), in which the initial state Φ_G represents the Fermi vacuum plus an electron-hole pair with momenta $\mathbf{p}_0 + \mathbf{q}_0$ and $\mathbf{p}_0$. Then all the preceding formulas remain valid except that, instead of (10), we have to substitute

$$\varkappa(\mathbf{p}) = \theta(\mathbf{p}) - \Delta(\mathbf{p} - \mathbf{p}_0) + \Delta(\mathbf{p} - \mathbf{p}_0 - \mathbf{q}_0),$$
$$\Delta(\mathbf{p}) = \begin{cases} 1, & \text{if } p = 0 \\ 0, & \text{if } p \neq 0. \end{cases} \tag{14}$$

The interaction Hamiltonian, consisting of operators η^* and η defined with respect to Φ_G, already contains processes of scattering of an electron (or hole) with absorption or emission of a pair with respect to the Fermi vacuum Φ_V. Thus the treatment of the new model Hamiltonian is equivalent to summation of diagrams of the type shown in Fig. 1 (except for the ground-state diagrams). Then the energy of excitation of a pair is given by the obvious relation:

$$\bar{E}(\mathbf{p}_0 + \mathbf{q}_0, \mathbf{p}_0) = E(\mathbf{p}_0 + \mathbf{q}_0, \mathbf{p}_0) - E_0 \tag{15}$$
$$= \omega(\mathbf{p}_0, \mathbf{q}_0) + W_x(\mathbf{p}_0, \mathbf{q}_0) + \Delta E(\Phi_G) - \Delta E(\Phi_V),$$

where $\bar{E}(\mathbf{p}_0 + \mathbf{q}_0, \mathbf{p}_0)$ is the excitation energy of a pair; $E(\mathbf{p}_0 + \mathbf{q}_0, \mathbf{p}_0)$ is the energy of the excited state containing one pair, and E_0 is the energy of the ground state. $W_X(\mathbf{p}_0, \mathbf{q}_0) = W_X(\mathbf{p}_0 + \mathbf{q}_0) - W_X(\mathbf{p}_0)$ is the exchange correction to the excitation energy of the pair. The expression for $W_X(\mathbf{p})$ is well known:

$$W_x(\mathbf{p}) = -\frac{1}{\pi a r_s}\left[2 + \frac{1-p^2}{p}\ln\frac{1+p}{1-p}\right].$$

Substituting expression (11) for $\Delta E(\Phi_V)$ and $\Delta E(\Phi_G)$ in (15), using (10) and (14), and dropping all terms which vanish as $\Omega \to \infty$, we get the expression for the excitation energy of a pair

$$\bar{E}(\mathbf{p}_0 + \mathbf{q}_0, \mathbf{p}_0)$$
$$= \omega(\mathbf{p}_0, \mathbf{q}_0) + W_x(\mathbf{p}_0, \mathbf{q}_0) + G(\mathbf{p}_0 + \mathbf{q}_0) - G(\mathbf{p}_0),$$

$$G(\mathbf{p}_0) = -\frac{2}{\pi^3}\frac{1}{2\pi i} \tag{16}$$
$$\times\left[\int\limits_{|\mathbf{p}_0+\mathbf{q}|>1}\frac{d^3q}{q^2}\int_\Gamma \frac{\left(1 - z\tanh^{-1}\frac{1}{z}\right)2\omega(\mathbf{p}_0, \mathbf{q})\,dz}{\left[q^2 + \frac{4r_s\alpha}{\pi}\left(1 - z\tanh^{-1}\frac{1}{z}\right)\right](z^2 - \omega^2(\mathbf{p}_0, \mathbf{q}))}\right.$$
$$\left. + \int\limits_{|\mathbf{p}_0-\mathbf{q}|<1}\frac{d^3q}{q^2}\int_\Gamma \frac{\left(1 - z\tanh^{-1}\frac{1}{z}\right)2\omega(-\mathbf{p}_0, \mathbf{q})\,dz}{\left[q^2 + \frac{4r_s\alpha}{\pi}\left(1 - z\tanh^{-1}\frac{1}{z}\right)\right](z^2 - \omega^2(-\mathbf{p}_0, \mathbf{q}))}\right].$$

Using the obvious symmetry property $\bar{E}(-\mathbf{q}_0 - \mathbf{p}_0, -\mathbf{p}_0) = \bar{E}(\mathbf{p}_0 + \mathbf{q}_0, \mathbf{p}_0)$, we can change (16)* to the more convenient form:

$$G(\mathbf{p}_0) = -\frac{2}{\pi^4}\int\frac{d^3q}{q^2}$$
$$\times\int\limits_{-\infty}^{\infty}\frac{\left(1 - u\tan^{-1}\frac{1}{u}\right)\omega(\mathbf{p}_0, \mathbf{q})\,du}{\left[q^2 + \frac{4r_s\alpha}{\pi}\left(1 - u\tan^{-1}\frac{1}{u}\right)\right](u^2 + \omega^2(\mathbf{p}_0, \mathbf{q}))}. \tag{17}$$

From (17) we see that $G(\mathbf{p}_0)$ is independent of the volume of the system, which should have been anticipated from physical considerations, since this quantity represents the correlation correction to the energies of the elementary excitations, and these quantities are intensive quantities as $\Omega \to \infty$. The explicit form of $G(\mathbf{p}_0)$ can be obtained by numerical integration of (17).

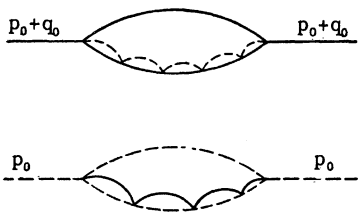

FIG. 1. We use diagrams of type considered in reference 3. The solid line shows the electron, the dotted line – the hole.

5. SPECIFIC-HEAT CORRECTION

From (16), we can easily find the correction to the specific heat in the Gell-Mann approximation.[5] For this purpose we need to know the derivative of $G(\mathbf{p}_0)$ at $|\mathbf{p}_0| = 1$ (from now on, we assume that $p_0 \gg 1$). Since the divergence in the perturbation series for the derivative of $G(\mathbf{p}_0)$ at $|\mathbf{p}_0| = 1$ appears one order earlier than for $G(\mathbf{p}_0)$, we need only include terms proportional to r_s^{-1} and $r_s^{-1}\ln r_s$.[5] In calculating the derivative of $G(\mathbf{p}_0)$

*In (16) and subsequent formulas, $\omega(\mathbf{p}_0, \mathbf{q}) = \mathbf{q}\cdot(2\mathbf{p}_0 + \mathbf{q})/2q$.

in this approximation, it is sufficient to differentiate (16) only with respect to the limit of the q integration, since the other terms give contributions of higher order in r_s. For the z integration it is convenient to choose the contour Γ as shown in Fig. 2. The integrals over Γ_1 and Γ_3 give no contribution, since the subsequent q integration goes only over the surface of the unit sphere $|\mathbf{q} - \mathbf{p}_0| = 1$ ($|\mathbf{p}_0| = 1$), on which $\omega(\mathbf{p}_0, \mathbf{q}) = 0$. Thus we have

$$\frac{dG(\mathbf{p}_0)}{dp_0}\Big|_{p_0=1} = \frac{-4}{\pi^3}\int \frac{d^3q}{q^2}\,\mathrm{res}\left[\frac{\left(1 - z\tanh^{-1}\frac{1}{z}\right)2\omega(\mathbf{p}_0, \mathbf{q})(\mathbf{p}_0, \mathbf{p}_0 + \mathbf{q})}{\left[q^2 + \frac{4r_s\alpha}{\pi}\left(1 - z\tanh^{-1}\frac{1}{z}\right)\right](z^2 - \omega^2(\mathbf{p}_0, \mathbf{q}))}\right]$$

$$\times\,\delta(1 - |\mathbf{p}_0 + \mathbf{q}|) = \frac{-4}{\pi^3}\int \frac{d^3q}{q^2}\left[\frac{\left(1 - z\tanh^{-1}\frac{1}{z}\right)(\mathbf{p}_0, \mathbf{p}_0 + \mathbf{q})}{q^2 + \frac{4r_s\alpha}{\pi}\left(1 - z\tanh^{-1}\frac{1}{z}\right)}\right]_{z=\omega(\mathbf{p}_0,\mathbf{q})} \delta(1 - |\mathbf{p}_0 + \mathbf{q}|)$$

(res denotes the residue at the point $z = \omega(\mathbf{p}_0, \mathbf{q})$). Noting that $\omega(\mathbf{p}_0, \mathbf{q}) = 0$ on the surface $|\mathbf{q} + \mathbf{p}_0| = 1$ ($|\mathbf{p}_0| = 1$), and also that $\lim_{z\to 0} z\tanh^{-1}(1/z) = 0$, we have

$$\frac{dG(\mathbf{p}_0)}{dp_0}\Big|_{p_0=1} = \frac{-4}{\pi^3}\int \frac{d^3q\,(\mathbf{p}_0, \mathbf{p}_0 + \mathbf{q})}{q^4(1 + 4r_s\alpha/\pi q^2)}\,\delta(1 - |\mathbf{p}_0 + \mathbf{q}|)$$

$$= \frac{-2}{\pi^2}\int_{-1}^{+1} \frac{x\,dx}{(1-x)^2(1 + 2r_s\alpha/\pi\,(1-x))}. \qquad (18)$$

Here we have changed the integral over $\mathbf{q}$ to a surface integral over a sphere with center at $\mathbf{p}_0$, and $x = \mathbf{p}_0 \cdot (\mathbf{p}_0 + \mathbf{q})$. Expression (18) diverges logarithmically. When we add to it the derivative of $W_X(\mathbf{p}_0)$ at $p_0 = 1$, the logarithmic divergences cancel and we get an expression which is identical with Gell-Mann's[5] result:

$$\frac{\partial\bar{E}(p_0)}{\partial p_0}\Big|_{p_0=1} \approx \frac{2}{\alpha^2 r_s^2} + \frac{1}{\pi r_s\alpha}\int_{-1}^{+1} \frac{x\,dx}{(1-x)}\left[1 + \frac{2\alpha r_s}{\pi}\frac{1}{(1-x)}\right]^{-1}.$$

For the specific heat, we find the expression

$$c/c_F = \left(1 + \frac{\alpha r_s}{2\pi}[-\ln r_s + \ln[\pi/2] - 2] + \cdots\right)^{-1}, \qquad (19)$$

where c and c_F are the specific heats of the electron gas in the presence and absence of interaction, respectively.

6. DISCUSSION OF RESULTS

From the results of Secs. 3 and 4 it follows that in an electron gas there are two types of elementary excitations: excitations of individual pairs and excitations of the collective type (plasma oscillations). In investigating the physical properties of the electron gas in a metal at ordinary temperatures, excitations of the individual type are of primary interest, since we know that the plasma oscillations are not thermally excited. The essential point is that, even when we take account quite precisely of the dynamical correlations between electrons, the main role in these phenomena is played by the independent elementary excitations: the electron-hole pairs (quasi-particles, satisfying Fermi statistics). The divergences in the usual perturbation theory expansion for a system with Coulomb interaction prove to be ephemeral, and after appropriate formal manipulation give finite contributions to the energy of the ground state and the elementary excitations. These ideas are realized in a very simple way in the present paper by using the method of approximate second quantization. We have obtained finite expressions for the self energy of the elementary excitations. Therefore, to a certain extent our results can be regarded as a justification of the independent particle model in the electron theory of metals, which has been applied with great success to treat a variety of metallic properties despite the fact that this model completely neglects correlation effects. This problem has been repeatedly discussed by various authors (cf. for example, reference 7). The treatment of a more realistic model, in which the effect of the periodic field of the crystal lattice is included right from the start, will be the subject of a separate investigation.

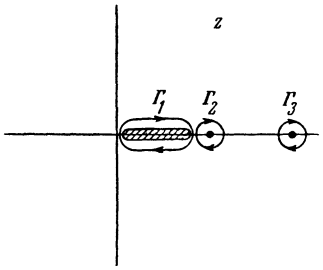

FIG. 2. The countour Γ is split into $\Gamma_1, \Gamma_2, \Gamma_3$. Γ_1 circles the cut from the origin to the point $z = 1$. Γ_2 and Γ_3 encircle the poles $z = \omega(\mathbf{p}_0, \mathbf{q})$ and $z = \omega_p(\mathbf{q})$, respectively.

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ON THE THEORY OF HIGH-SPIN PARTICLES

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An algebraic method is suggested for treating the relativistically invariant equations of high-spin particles. The direct product of generalized Dirac algebras underlies the analysis. This method can be used to obtain explicit expressions for the infinitesimal rotation matrix, the spin operator, the metric and reflection operator, as well as to limit the number of representations taken into account. The equations can be treated directly either in tensor or in spin-tensor form. The commutation relations are automatically obtained in parametric form.

1. INTRODUCTION

We consider relativistically invariant equations of the form

$$\alpha_k \partial_k \psi + \varkappa \psi = 0, \tag{1}$$

where ψ is a particle wave function with a finite number of components which transforms according to some finite-dimensional representation of the Lorentz group. Equation (1) shall be called relativistically invariant if under the Lorentz transformation $x'_i = l_{ik} x_k$ together with the transformation $\psi' = S\psi$ it remains formally invariant, in other words if

$$\alpha_i = l_{ij} S \alpha_j S^{-1}. \tag{2}$$

Now the conditions of (2) are fulfilled if

$$[I_{ik}, I_{jl}] = -\delta_{ij} I_{kl} + \delta_{il} I_{kj} + \delta_{kj} I_{il} - \delta_{kl} I_{ij}, \tag{3}$$

$$[\alpha_i, I_{jk}] = \delta_{ij} \alpha_k - \delta_{ik} \alpha_j, \tag{4}$$

where the I_{ij} are the infinitesimal-rotation matrices, defined by

$$\psi' = \psi + {}^1/_2 \varepsilon_{ik} I_{ki} \psi.$$

We shall consider here the problem of finding all relativistically-invariant equations (1) that satisfy the additional requirements that they be invariant under reflection, that there exist a nondegenerate real Lagrangian, that the energy density be positive definite for particles with integral spin or the charge be positive definite for particles with half-integral spin, and that the equations be irreducible.

This problem has been treated in general form by Gel'fand and Iaglom.[1] Their method can be used, in principle, to obtain all possible equations for high-spin particles. In actually obtaining the equations, however, certain difficulties arise. Among these are, in particular, the analysis of irreducibility, the transition to spin-tensor (or tensor) form, and the determination of the algebra of the α_k matrices. Further, there is no way to tell whether all of the equations belonging to a given maximum spin have been found.

While Gel'fand and Iaglom base their considerations on explicit expressions for the infinitesimal operators, Harish-Chandra[2,3] develops an algebraic method based on the study of an algebra designated $U(\alpha)$. This is the α-matrix algebra of the forms