ON THE SOLUTION OF THE KINETIC EQUATIONS OF TRANSPORT OF NEUTRONS OR γ-RAY QUANTA BY THE METHOD OF PARTIAL PROBABILITIES

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The problem of solving the kinetic equations for the slowing down and diffusion of neutrons and for the propagation of γ-ray quanta is reduced to a less complicated problem of multiple integration. An exact solution of the kinetic equation (stationary and also nonstationary) is found in the form of a sum, in which the κ-th term is an approximately 3κ-fold integral, which has as its meaning the probability for the transition of a particle from one point of its phase space to another after κ collisions. In the particular case of the slowing down and diffusion of neutrons with constant mean free path, it is shown that this exact solution, which depends on all six variables (the three space coordinates and the three components of the momentum) reduces to simple quadratures and sums.

1. STATEMENT OF THE PROBLEM

THE stationary kinetic equation for the slowing down of neutrons is (cf. reference 1):

$$[l(u)\,\boldsymbol{\Omega}\,\mathrm{grad}+1]\,\Psi(\mathbf{r},\boldsymbol{\Omega},u)=\int_0^u du_1\int d^2\Omega_1\Psi(\mathbf{r},\boldsymbol{\Omega}_1 u_1)\times f(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_1,u-u_1)\,h(u_1)+Q(\mathbf{r},\boldsymbol{\Omega},u). \quad (1)$$

Here $\Psi(\mathbf{r},\boldsymbol{\Omega},u)$ is the number of collisions per unit volume of the phase space at the element $d^3\mathbf{r}d^2\boldsymbol{\Omega}du$; $u=\ln(E/E_0)$, with E the energy of the neutron and E_0 a certain energy taken as a unit; $\boldsymbol{\Omega}$ is the unit vector of the direction of the momentum; $h(u)=l(u)/l_S(u)$, with l the mean free path and l_S the mean free path against scattering; $f(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_1,u)$ is the scattering function, which in the case of symmetry about the center of mass is given by

$$\sum_M C_M\frac{M+1}{8\pi M}e^{-u}\delta\left\{\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_1-\left[\frac{M+1}{2}e^{-u/2}-\frac{M-1}{2}e^{u/2}\right]\right\},$$

where M is the mass of the nucleus in units of the neutron mass; l_{SM} is the free path against scattering by the element of mass M.

It is required to find the Green's function G of this equation, i.e., a function such that

$$\Psi(\mathbf{r},\boldsymbol{\Omega},u)=\int d^3r_1d^2\Omega_1du_1G(\mathbf{r}_1,\mathbf{r},\boldsymbol{\Omega}_1,\boldsymbol{\Omega},u_1,u)\,Q(\mathbf{r}_1,\boldsymbol{\Omega}_1,u_1). \quad (2)$$

The integration is taken over the entire six-dimensional phase space. Substituting Eq. (2) into Eq. (1), one verifies without difficulty that G satisfies the following equation:

$$[l\boldsymbol{\Omega}\,\mathrm{grad}+1]\,G(\mathbf{r}_1,\mathbf{r},\boldsymbol{\Omega}_1,\boldsymbol{\Omega},u_1,u)=\int_0^u du_2\int d^2\Omega_2G(\mathbf{r}_1,\mathbf{r},\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2,u_1,u_2)\times f(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_2,u-u_2)+\delta(\mathbf{r}_1-\mathbf{r})\,\delta(\boldsymbol{\Omega}_1-\boldsymbol{\Omega})\,\delta(u_1-u). \quad (3)$$

In obtaining this equation one must change the order of integration of the variables with indices 1 and 2.

Equation (3) is an integro-differential equation. It can be transformed into the purely integral equation

$$G(\mathbf{r}_1,\mathbf{r},\boldsymbol{\Omega}_1,\boldsymbol{\Omega},u_1,u)=G_0(\mathbf{r}_1,\mathbf{r},\boldsymbol{\Omega}_1,\boldsymbol{\Omega},u_1,u)+\int du_2\int_0^{u_2}du_3\int d^2\Omega_2\int d^2\Omega_3\int d^3r_2G_0(\mathbf{r}_2,\mathbf{r},\boldsymbol{\Omega}_2,\boldsymbol{\Omega},u_2,u)\times G(\mathbf{r}_1,\mathbf{r}_2,\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_3,u_1,u_3)\,f(\boldsymbol{\Omega}_2\cdot\boldsymbol{\Omega}_3,u_2-u_3)\,h(u_3), \quad (4)$$

where $G_0(\mathbf{r}_1,\mathbf{r},\boldsymbol{\Omega}_1,\boldsymbol{\Omega},u_1,u)$ satisfies Eq. (3) without the integral term in the right member

$$(l\boldsymbol{\Omega}\,\mathrm{grad}+1)\,G_0(\mathbf{r}_1,\mathbf{r},\boldsymbol{\Omega}_1,\boldsymbol{\Omega},u_1,u)=\delta(\mathbf{r}_1-\mathbf{r})\,\delta(\boldsymbol{\Omega}_1-\boldsymbol{\Omega})\,\delta(u_1-u). \quad (5)$$

For the proof one must substitute the expression for G in the right member of Eq. (4) into the left member of Eq. (3) and note that the gradient operator acts only on the coordinates without indices.

Equations (5) and (4) can be given a simple physical interpretation. The quantity G_0 is proportional to the probability of transition of a neutron with the momentum direction $\mathbf{\Omega}_1$ and energy corresponding to u_1 from the point $\mathbf{r}_1$ to the point $\mathbf{r}$, with its momentum direction changed to $\mathbf{\Omega}$ and its energy variable to u, without collisions. Stated more simply: the neutron has passed from point $\vec{1}$ of phase space to point $\vec{0}$ without scattering or absorption $[G_0(\vec{1}, \vec{0})]$.

The quantity $G(\vec{1}, \vec{0})$ is proportional to the probability (hereafter we shall simply say is the probability) of transition of the neutron from the point $\vec{1}$ to the point $\vec{0}$ with the occurrence of any number of collisions. According to Eq. (4) this probability is equal to the sum of the probabilities of the following events: 1. The neutron has passed from $\vec{1}$ to $\vec{0}$ without having any collisions (the first term G_0), 2. The neutron, having come from $\vec{1}$, and after having any number of collisions, was at the point $\mathbf{r}_2$ with momentum direction $\mathbf{\Omega}_3$ and energy variable u_3 [factor $G(\mathbf{r}_1, \mathbf{r}_2, \mathbf{\Omega}_1, \mathbf{\Omega}_3, u_1, u_3)$]. At this point it was scattered or absorbed, with transition to the point $\vec{2}$ [factor $h(u_3) f(\mathbf{\Omega}_2\mathbf{\Omega}_3, u_2 - u_3)$]. After this it has passed without collisions from the point $\vec{2}$ to $\vec{0}$ [factor $G_0(\vec{2}, \vec{0})$]. The integration is taken over all the intermediate-state variables $\mathbf{r}_2, \mathbf{\Omega}_2, \mathbf{\Omega}_3, u_2, u_3$. The function f assures that the laws governing the scattering are satisfied.

Equation (4) is thus a recussion formula that relates the probability distribution after each κ-th collision with the $(\kappa + 1)$-th one.

2. THE EXPRESSION FOR THE GREEN'S FUNCTION OF THE KINETIC EQUATION IN TERMS OF MULTIPLE INTEGRALS

To solve the integral equation (4) we shall employ the usual method of successive approximations. This makes it possible to find successively the pictures after $0, 1, 2, \ldots \kappa$ collisions (functions $G_0, G_1, \ldots, G_\kappa$, respectively). Each subsequent $G_{\kappa+1}$ is found by integrating the product of the preceding function by the kernel G_0. G_κ can be called a partial probability, since it plays the same role for the kinetic equation that a partial wave does for the wave equation.

Let us first find G_0 from Eq. (5). Inserting $\delta(\mathbf{r}_1 - \mathbf{r})$ in the form

$$\delta(\mathbf{r}_1 - \mathbf{r}) = \frac{1}{8\pi^3}\int d^3\mathbf{k} \exp(-i\mathbf{k}(\mathbf{r}_1 - \mathbf{r})),$$

and dividing by the operator $(l\mathbf{\Omega}\,\mathrm{grad} + 1)$, we get

$$G_0(\vec{1}, \vec{0}) = \delta(\mathbf{\Omega}_1 - \mathbf{\Omega})\,\delta(u_1 - u)\,\frac{1}{8\pi^3}\int \frac{\exp(-i\mathbf{k}(\mathbf{r}_1 - \mathbf{r}))}{1 + il\mathbf{k}\cdot\mathbf{\Omega}}\,d^3\mathbf{k}\,. \quad (6)$$

Performing the integration with respect to one of the components of the vector $\mathbf{k}$ (for example, k_x) by means of the theory of residues, we get

$$\begin{gathered} G_0(\vec{1}, \vec{0}) = \frac{1}{l(u)}\delta(\mathbf{\Omega}_1 - \mathbf{\Omega})\,\delta(u_1 - u)\,\frac{\gamma[(r_{1x} - r_x)/\Omega_x]}{|\Omega_x|} \\ \times \exp\left(-\frac{r_{1x} - r_x}{l\Omega_x}\right)\delta\left(\Omega_y\frac{r_{1x} - r_x}{\Omega_x} - r_{1y} + r_y\right) \\ \times\,\delta\left(\Omega_z\frac{r_{1x} - r_x}{\Omega_x} - r_{1z} + r_z\right); \\ \gamma(\alpha) = \begin{cases} 1 & \text{for } \alpha > 0, \\ 0 & \text{for } \alpha < 0. \end{cases} \end{gathered} \quad (7)$$

From this it can be seen that G_0 has a singularity on the straight line

$$\frac{r_{1x} - r_x}{\Omega_x} = \frac{r_{1y} - r_y}{\Omega_y} = \frac{r_{1z} - r_z}{\Omega_z} = \mathrm{const} = s > 0. \quad (8)$$

According to Eq. (8) the vectors $(\mathbf{r} - \mathbf{r}_1)$ and $\mathbf{\Omega}$ are strictly parallel. Since $|\mathbf{\Omega}| = 1$, this means that

$$s = |\mathbf{r}_1 - \mathbf{r}|. \quad (9)$$

Equation (7) can then be rewritten in the form

$$\begin{gathered} G_0(\vec{1}, \vec{0}) = \frac{\exp(l^{-1}|\mathbf{r}_1 - \mathbf{r}|)}{l\,|\mathbf{r}_1 - \mathbf{r}|^2}\,\delta(\mathbf{\Omega}_1 - \mathbf{\Omega}) \\ \times\,\delta(u_1 - u)\,\delta\left(\mathbf{\Omega} - \frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{r}|}\right). \end{gathered} \quad (10)$$

In these transformations repeated use has been made of the property of the δ function,

$$\delta[f(x)] = \sum_s f(x - x_s)/f'(x_s); \quad f(x_s) = 0. \quad (11)$$

It can be seen from Eq. (10) that G_0 contains a dependence of the form $R^{-2}\exp(R/l)$, which is characteristic of a transmitted beam, as was naturally to be expected.

It must be remembered that a δ function of unit vectors factors into only two δ functions, corresponding to the number of independent components. For example,

$$\delta(\mathbf{\Omega} - \mathbf{\Omega}_1) = \delta(\mu - \mu_1)\,\delta(\varphi - \varphi_1),$$

where μ, μ_1 are the cosines of the polar angles and φ, φ_1 are the azimuthal angles.

To solve Eq. (4) it is more convenient to insert in it the function G_0 in the form (6), not in the form (10). We obtain:

$$G(\vec{1},\vec{0}) = G_0(\vec{1},\vec{0}) + \frac{1}{8\pi^3}\int_0^u du_3 \int d^2\mathbf{\Omega}_3 \int d^3\mathbf{r}_2 h(u_3)\; G(\mathbf{r}_1,\mathbf{r}_2,\mathbf{\Omega}_1,\mathbf{\Omega}_3,u_1,u_3) f(\mathbf{\Omega}\cdot\mathbf{\Omega}_3, u-u_3)\int d^3\mathbf{k}\, \frac{\exp[-i\mathbf{k}(\mathbf{r}_2-\mathbf{r})]}{1+il\mathbf{k}\cdot\mathbf{\Omega}}. \quad (12)$$

Carrying out a straightforward calculation, we find:

$$G_1 = \frac{1}{8\pi^3}\int d^3\mathbf{k}\, \frac{f(\Omega\cdot\Omega_1, u-u_1)\exp[-i\mathbf{k}(\mathbf{r}_1-\mathbf{r})]\, h(u_1)}{(1+il\mathbf{k}\cdot\Omega)(1+il_1\mathbf{k}\cdot\Omega_1)}.$$

Here and in what follows use is made of the fact that the integration with respect to $\mathbf{r}_2$ can be performed at once and gives $\delta(\mathbf{k}_1-\mathbf{k})$, so that the integral with respect to $\mathbf{k}$ will still remain a triple integral. Furthermore, we have written as a simplification

$$l_n = l(u_n);\; l = l(u).$$

Continuing the process of calculating the "partial probabilities," we find

$$G_2 = \frac{1}{8\pi^3}\int d^3\mathbf{k}\int_0^u du_2 \int d^2\mathbf{\Omega}_1\, \frac{\exp[-i\mathbf{k}(\mathbf{r}_1-\mathbf{r})]\, h(u_1)\, h(u_2)}{(1+il\mathbf{k}\cdot\Omega)(1+il_1\mathbf{k}\cdot\Omega_1)(1+il_2\mathbf{k}\cdot\Omega_2)}$$

$$\times f(\mathbf{\Omega}_2\cdot\mathbf{\Omega}_1, u_2-u_1)\, f(\mathbf{\Omega}\cdot\mathbf{\Omega}_2, u-u_2);$$

$$G_\varkappa = \frac{1}{8\pi^3}\int d^3\mathbf{k}\int_0^u du_2 \int d^2\mathbf{\Omega}_2 \int_0^{u_2} du_3 \int d^2\mathbf{\Omega}_3 \ldots \int_0^{u_{\varkappa-1}} du_\varkappa \int d^2\mathbf{\Omega}_\varkappa$$

$$\times \exp[-i\mathbf{k}(\mathbf{r}_1-\mathbf{r})]\frac{h(u_1)\, f(\Omega\cdot\Omega_2, u-u_2)\, f(\Omega_\varkappa\cdot\Omega_1, u_\varkappa-u_1)}{(1+il\mathbf{k}\cdot\Omega)(1+il_1\mathbf{k}\cdot\Omega_1)}$$

$$\times \prod_{\alpha=2}^{\varkappa}\frac{h(u_\alpha)}{1+il_\alpha\mathbf{k}\cdot\Omega_\alpha}\prod_{\omega=2}^{\omega=\varkappa-1} f(\mathbf{\Omega}_\omega\cdot\mathbf{\Omega}_{\omega+1},\; u_\omega-u_{\omega+1}),$$

$$G(\vec{1},\vec{0}) = \sum_{\varkappa=0}^{\infty} G_\varkappa(\vec{1},\vec{0}). \quad (13)$$

This is indeed the final form for the Green's function of the stationary kinetic equation. We note that nowhere in the above calculations has use been made of the concrete form of the function $f(\mathbf{\Omega}\cdot\mathbf{\Omega}_1, u-u_1)$, nor of its dependence on any combination of its arguments. Therefore the formulas (13) are also valid when $f \equiv f(\mathbf{\Omega}, \mathbf{\Omega}_1, u, u_1)$.

Each term of G is positive, according to its physical meaning, so that the series (13) is a series of positive terms. Thus the problem of solving the kinetic equation has been reduced to a problem of multiple integration. Well developed procedures of approximate integration exist for the calculation of multiple integrals. We once again emphasize the fact that Eq. (13) provides an exact solution of the kinetic equation, valid for arbitrary energies and distances from the source, and for any form of the function f.

3. GENERALIZATION TO THE CASE OF THE NONSTATIONARY KINETIC EQUATION

In this case the operator $(l\mathbf{\Omega}\,\mathrm{grad}+1)$ in Eq. (1) is replaced by

$$\left(\frac{l(u)}{v}\frac{\partial}{\partial t} + l(u)\,\mathbf{\Omega}\,\mathrm{grad} + 1\right).$$

The integral equation for the Green's function takes the form

$$G(t_1-t,\vec{1},\vec{0}) = G_0(t_1-t,\vec{1},\vec{0}) + \int du_2\int_0^{u_2} du_3 \int d^2\mathbf{\Omega}_2\int d^2\mathbf{\Omega}_3$$

$$\times \int dt_2\int d^3\mathbf{r}_2 G_0(t_2-t,\vec{2},\vec{0})G(t_1-t_2,\mathbf{r}_1,\mathbf{r}_2,\mathbf{\Omega}_1,\mathbf{\Omega}_3,u_1,u_3)$$

$$\times f(\mathbf{\Omega}_2\cdot\mathbf{\Omega}_3, u_2-u_3)\, h(u_3), \quad (14)$$

where G_0 satisfies the equation

$$\left(\frac{l}{v}\frac{\partial}{\partial t} + l\mathbf{\Omega}\,\mathrm{grad} + 1\right)G_0(t_1-t,\vec{1},\vec{0})$$
$$= \delta(t_1-t)\,\delta(\mathbf{r}_1-\mathbf{r})\,\delta(\mathbf{\Omega}_1-\mathbf{\Omega})\,\delta(u_1-u). \quad (15)$$

Representing δ functions as Fourier integrals, we get

$$G_0(t_1-t,\vec{1},\vec{0}) = \frac{1}{16\pi^4}\delta(\mathbf{\Omega}_1-\mathbf{\Omega})\;\delta(u_1-u)\int d^3\mathbf{k}\,d\sigma$$

$$\times \frac{\exp[-i\sigma(t_1-t)-i\mathbf{k}(\mathbf{r}_1-\mathbf{r})]}{1+il\sigma\tau+il\mathbf{k}\cdot\Omega} = \frac{1}{l\tau}\delta(\mathbf{\Omega}_1-\mathbf{\Omega})\,\delta(u_1-u)$$

$$\times\,\delta\left(\mathbf{r}_1-\mathbf{r}-\frac{t_1-t}{\tau}\mathbf{\Omega}\right)\exp\left(-\frac{|t_1-t|}{l\tau}\right), \quad (16)$$

$$\tau = l(u)/v.$$

Equation (16) expresses the physically obvious fact that G_0 is different from zero only on a segment of a straight line for which the equations are

$$\frac{r_{1x}-r_x}{\Omega_x} = \frac{r_{1y}-r_y}{\Omega_y} = \frac{r_{1z}-r_z}{\Omega_z} = |\mathbf{r}_1-\mathbf{r}| = \frac{|t_1-t|}{\tau}.$$

As before, the vector $\mathbf{\Omega}$ is parallel to the vector $\mathbf{r}_1-\mathbf{r}$, but the length of the segment depends on the time.

The method of successive approximations gives at once

$$G_\varkappa = \frac{1}{16\pi^4}\int d^3\mathbf{k}\,d\sigma\int_0^u du_2\int d^2\mathbf{\Omega}_2\int_0^{u_2} du_3\int d^2\mathbf{\Omega}_3\ldots\int_0^{u_{\varkappa-1}} du_\varkappa\int d^2\mathbf{\Omega}_x\, \frac{\exp[-i\sigma(t_1-t)-i\mathbf{k}(\mathbf{r}_1-\mathbf{r})]\, h(u_1)\, f(\Omega\cdot\Omega_2, u-u_2)\, f(\Omega_\varkappa\cdot\Omega_1, u_\varkappa-u_1)}{(1+il\sigma\tau+il\mathbf{k}\cdot\Omega)(1+il_1\sigma\tau_1+il_1\mathbf{k}\cdot\Omega_1)}$$

$$\times\prod_{\alpha=2}^{\varkappa}\frac{h(u_\alpha)}{1+il_\alpha\sigma\tau_\alpha+il_\alpha\mathbf{k}\cdot\Omega_\alpha}\prod_{\omega=2}^{\varkappa=1} f(\mathbf{\Omega}_\omega\cdot\mathbf{\Omega}_{\omega+1},\; u_\omega-u_{\omega-1});$$

$$G(t_1-t,\vec{1},\vec{0}) = \sum_{\varkappa=0}^{\infty} G_\varkappa(t_1-t,\vec{1},\vec{0}). \quad (17)$$

4. SOLUTION OF THE KINETIC EQUATION FOR THE PROPAGATION OF GAMMA-RAY QUANTA AND THE DIFFUSION OF THERMAL NEUTRONS

The transport equation for γ-ray quanta is

$$(l_\kappa \Omega \operatorname{grad} + 1)\Psi(\mathbf{r}, \Omega, E) = \int_E^{E_0} dE_1 \int d^2\Omega_1 h(E_1)$$

$$\times M(\Omega\cdot\Omega_1, E, E_1)\Psi(\mathbf{r}, \Omega_1, E_1) + Q(\mathbf{r}, \Omega, E), \qquad (18)$$

where $\Psi(\mathbf{r}, \Omega, E)/l_c$ is the number of quanta at the phase point $\vec{0}(\mathbf{r}, \Omega, E)$. The energy is expressed in units of the rest mass of the electron. l_c is the mean free path against Compton scattering, l_p is that against the photoelectric effect,

$$h = l_p l_c/(l_c + l_p),$$

and $M(\Omega\cdot\Omega_1, E, E_1)$ is the Klein-Nishina differential cross section for scattering of a quantum from the state Ω_1, E_1 into the state Ω, E.

The analogy between Eq. (18) and Eq. (1) is almost complete. Therefore the Green's function of Eq. (18) is given by Eqs. (10) and (13), if one makes in them the replacements

$$l \to l_c;\ u = 0 \to E_0;\ u \to E;\ f \to M;\ \int_0^{u_\alpha} \to \int_{E_\alpha}^{E_0}. \qquad (19)$$

It is also easy to write down the Green's function for the diffusion of thermal neutrons. In this case the kinetic equation does not contain the energy (cf. reference 2). Therefore in the formulas (13) one must omit all the integrations over u_α, so that, for example,

$$G_\varkappa^{\mathrm{dif}}(\vec{1}, \vec{0}) = \frac{1}{8\pi^3}\int d^3\mathbf{k}\int d^2\Omega_2 \int d^2\Omega_3 \ldots \int d^2\Omega_\varkappa$$

$$\times \frac{h^\varkappa f(\Omega\cdot\Omega_2) f(\Omega_\varkappa\cdot\Omega_1)}{(1 + il\mathbf{k}\cdot\Omega)(1 + il\mathbf{k}\cdot\Omega_1)(1 + il\mathbf{k}\cdot\Omega_\varkappa)} \prod_{\alpha=2}^{\varkappa-1} \frac{f(\Omega_\alpha\cdot\Omega_{\alpha+1})}{1 + il\mathbf{k}\cdot\Omega_\alpha}. \qquad (20)$$

5. CASE OF SLOWING DOWN AND DIFFUSION OF NEUTRONS IN A MEDIUM IN WHICH THE MEAN FREE PATH IS CONSTANT

The method of solving the kinetic equation by means of partial probabilities, which has been explained and applied to all the equations of transport of neutrons and γ-ray quanta in Secs. 1 to 4, has reduced the problem of obtaining the solutions of these equations to problems of multiple integration. For these latter problems there exists well-developed procedures for both exact and approximate solution.

As an example, in the present section we shall find the exact spatial and energetic distribution of neutrons slowed down in a medium in which the mean free path is constant, and in which the nuclei scatter the neutrons symmetrically in the center-of-mass system. The case of the diffusion of thermal neutrons is obtained as a special case of this one. As is well known, no method hitherto presented has made it possible to find the exact solution of the kinetic equation for this case. In performing the multiple integrations we shall employ the usual operator method and expansion in series of Legendre polynomials.

We expand all the scattering functions in series of Legendre polynomials

$$f(\Omega\cdot\Omega_1, u - u_1) = \sum A_a(u - u_1) P_a(\Omega\cdot\Omega_1) \qquad (21)$$

$$= \sum_{a=0}^{\infty}\sum_{m=0}^{a} \varepsilon_m \frac{(a-m)!}{(a+m)!} A_a(u - u_1) P_a^m(\mu) P_a^m(\mu_1)\cos m(\varphi - \varphi_1);$$

$$\varepsilon_m = \begin{cases} 1 \text{ for } m = 0, \\ 2 \text{ for } m \neq 0; \end{cases}$$

$P_a^m(\mu)$ are the associated Legendre polynomials; $\mu_\alpha = \mathbf{k}_0\cdot\Omega_\alpha$ is the cosine of the angle between the vectors Ω_α and $\mathbf{k}$; $\mathbf{k}_0 = \mathbf{k}/|\mathbf{k}|$; φ is the azimuthal angle of Ω_α in the plane perpendicular to $\mathbf{k}$.

This expansion at once makes it possible to simplify the multiple integration with respect to the logarithmic energy variables in the case in which l does not depend on u and either there is no capture of the fast neutrons $[h(u) \equiv 1]$, or else $h(u) = \mathrm{const}$.

In fact, we write

$$W = \int_0^u du_2 \int_0^{u_2} du_3 \ldots \int_0^{u_{\varkappa-1}} du_\varkappa A_a(u - u_2) A_b(u_2 - u_3)\ldots$$

$$\ldots A_g(u_{\varkappa-1} - u_\varkappa) A_h(u_\varkappa - u_1). \qquad (22)$$

We perform the Laplace transformation

$$W(p) = \int_0^\infty e^{-pu} W du.$$

Changing the order of integration over u and u_2, we get

$$W(p) = \int_0^\infty du_2 \int_{u_2}^\infty du e^{-pu} \int_0^{u_2} du_3 \ldots$$

$$\int_0^{u_{\varkappa-1}} du_\varkappa A_a(u - u_2)\ldots A_h(u_\varkappa - u_1)$$

$$= \int_0^\infty A_a(t_2) e^{-pt_2} dt_2 \int_0^\infty du_2 e^{-pu_2} \int_0^u du_3 \ldots \int_0^{u_{\varkappa-1}} du_\varkappa$$

$$\times A_b(u_2 - u_3)\ldots A_h(u_\varkappa - u_1).$$

Continuing this process in the same way and writing

$$A_a(p) = \int_0^\infty e^{-pu} A_a(u)\,du, \tag{23}$$

we get

$$W(p) = A_a(p)\,A_b(p)\ldots A_g(p)\int_0^\infty e^{-pu_\varkappa} A_h(u_\varkappa - u_1)\,du_\varkappa$$
$$= A_a(p)\,A_b(p)\ldots A_g(p)\,e^{-pu_1} A_h(p). \tag{24}$$

Thus instead of the multiple integrals with respect to the u_α we have a single integration:

$$W = L^{-1}(p, u - u_1)\,A_a(p)\,A_b(p)\ldots A_g(p)\,A_h(p),$$

$$L^{-1}(p, u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pu}dp, \tag{25}$$

where, as usual, the straight line $\mathrm{Re}(p) = c$ is drawn to the right of the singular points.

To avoid misunderstanding we remark that just as in the kinetic equations, the integration over u starts at zero in Eq. (23), too. This means that the maximum energy of the neutrons from the sources is taken to correspond to zero (in the opposite case the integration would be taken from $-\infty$). Therefore we always have

$$A_a(u) \equiv 0 \text{ for } u < 0.$$

We now go on to the integration over the solid angles. After the series expansion and the Laplace transformation, there remain integrals and sums of the form

$$\sum_{g,h=0}^{\infty}\sum_{m=0}^{g}\sum_{n=0}^{h} \varepsilon_m\varepsilon_n A_g^m(p)\,A_h^n(p)\int_0^{2\pi} d\varphi_\varkappa \int_{-1}^{+1} d\mu_\varkappa \cos m(\varphi_{\varkappa-1} - \varphi_\varkappa)\cos n(\varphi_\varkappa - \varphi_1)\,\frac{P_g^m(\mu_{\varkappa-1})P_g^m(\mu_\varkappa)\,P_h^m(\mu_\varkappa)\,P_h^m(\mu_1)}{1 + ilk\mu_\varkappa}$$

$$= \sum_{g,h=0}^{\infty}\sum_{m=0}^{\min(g,h)} \frac{4\pi i}{lk}\varepsilon_m A_g^m(p)\,A_h^m(p)\,P_g^m(\mu_{\varkappa-1})\,D_{gh}^m\left(\frac{i}{kl}\right)P_h^m(\mu_1)\cos m(\varphi_{\varkappa-1} - \varphi_1); \tag{26}$$

where the quantities

$$D_{gh}^m(y) = D_{hg}^m(y) = \begin{cases} P_g^m(y)\,Q_h^m(y) & \text{for } g \leqslant h, \\ Q_g^m(y)\,P_h^m(y) & \text{for } g \geqslant h \end{cases}$$

are calculated in the Appendix. Here we have written

$$A_g^m(p) = A_g(p)\,(g - m)!\,/\,(g + m)! \tag{27}$$

Carrying out successively the integrations over $\mu_{K-1}, \mu_{K-2}, \ldots \mu_2$, we get:

$$G_\varkappa(\vec{1}, \vec{0}) = \frac{1}{8\pi^3}\sum_{a,b,\ldots,h=0}^{\infty}\sum_{m=0}^{\min(a,b,\ldots,h)} L^{-1}(p, u - u_1)\int d^3\mathbf{k}\exp\left[-i\mathbf{k}(\mathbf{r}_1 - \mathbf{r})\right]\varepsilon_m A_a^m(p)\,A_b^m(p)\ldots A_g^m(p)\,A_h^m(p)$$

$$\times\left(\frac{4\pi i}{lk}\right)^{\varkappa-1}\frac{P_a^m(\mathbf{k}_0\cdot\Omega)D_{ab}^m D_{bc}^m\ldots D_{gh}^m(i/lk)\,P_h^m(\mathbf{k}_0\cdot\Omega_1)}{(1 + il\mathbf{k}\cdot\Omega)(1 + il\mathbf{k}\cdot\Omega_1)}\cos m(\varphi - \varphi_1) \tag{28}$$

It must be remembered that the angles φ and φ_1 are measured in the plane perpendicular to $\mathbf{k}$, so that $\varphi - \varphi_1$ is the angle between $\mathbf{k}\times\mathbf{\Omega}$ and $\mathbf{k}\times\mathbf{\Omega}_1$.

The expression (28) can be considerably simplified if we expand the fractional quantities depending on $\mathbf{\Omega}$ and $\mathbf{\Omega}_1$ in series of $P_c(\mathbf{k}_0\cdot\mathbf{\Omega})$ and $P_d(\mathbf{k}_0\cdot\mathbf{\Omega}_1)$, interchange the summations over the upper and lower indices, and introduce the matrix notation:

$\|AD^m\|$ — is the matrix of the quantities $\sqrt{A_a^m(p)\,A_b^m(p)}\,D_{ab}^m(i/kl)$.

Then

$$G_\varkappa(\vec{1}, \vec{0}) = \sum_{m=0}^{\infty}\sum_{a=m}^{\infty}\sum_{b=m}^{\infty}\frac{\varepsilon_m}{4(2\pi)^5}L^{-1}(p, u - u_1)\int d^3\mathbf{k}\ \exp\left[-i\mathbf{k}(\mathbf{r}_1 - \mathbf{r})\right]\frac{(2a+1)(2b+1)}{\sqrt{A_a^m(p)\,A_b^m(p)}}\,P_a^m(\mathbf{k}_0\cdot\mathbf{\Omega})\,P_b^{-m}(\mathbf{k}_0\cdot\mathbf{\Omega}_1)$$

$$\times\left(\frac{4\pi i}{lk}\right)^{\varkappa+1}\left(\|AD^m\|^{\varkappa+1}\right)_{ab}\cos m(\varphi_1 - \varphi). \tag{29}$$

Summing all the G_K and using the property of the geometric progression, we get

$$G(\vec{1},\vec{0})=\sum_{\varkappa=0}^{\infty}G_\varkappa=\sum_{m=0}^{\infty}\sum_{a,b=m}^{\infty}\frac{i\varepsilon_m(2a+1)(2b+1)}{2(2\pi)^4}L^{-1}(p,u-u_1)$$

$$\times\int d^3\mathbf{k}\exp[-i\mathbf{k}(\mathbf{r}_1-\mathbf{r})]\frac{\Phi_{ab}^m(p,k,l)P_a^m(\mathbf{k}_0\cdot\Omega)P_b^{-m}(\mathbf{k}_0\cdot\Omega_1)\cos m(\varphi-\varphi_1)}{lk(A_a^m(p)A_b^m(p))^{1/2}};\qquad(30)$$

$$\Phi_{ab}^m(p,k,l)=\left(\|AD^m\|/\left(1-\frac{4\pi i}{lk}\|AD^m\|\right)\right)_{ab}.\qquad(31)$$

Here the matrix of the fraction must be understood in the sense of a series expansion. In addition we have used the fact that Eq. (29) is also valid for $\kappa=0$. To prove this one expands the expression for G_0 obtained from Eq. (16) in a Laplace integral and Legendre series.

We now recall that in the particular case of scattering symmetrical in the center-of-mass system one has the formulas (cf. reference 1)

$$A_0(p)=\sum_M C_M\frac{(M+1)^2}{16\pi M}\frac{1-\exp[-(p+1)u_M]}{p+1};$$

$$A_1(p)=\sum_M C_M\frac{3(M+1)^2}{4\pi M}$$

$$\times\frac{2p+2-M+(2p+2+M)\exp[-(p+1)u_M]}{(2p+1)(2p+3)};\ \ldots;$$

$$A_a(p)=\sum_M C_M\frac{(2a+1)(M+1)^2}{16\pi M}\int_0^{u_M}e^{-(p+1)u}P_a(X_M(u))\,du;$$

$$X_M(u)=\frac{M+1}{2}e^{-u/2}-\frac{M-1}{2}e^{u/2};\qquad u_M=2\ln\frac{M+1}{M-1}.$$

Thus Eqs. (30) and (31) give the Green's function of the kinetic equation for the slowing down of neutrons. They are of simple structure and are written in vector form. We note that the functions G_K given by Eq. (29) are interesting in themselves, since they are the probabilities for κ collisions. In particular, it will be shown below that in the case of atoms of a single kind

$$G_\varkappa\equiv 0\ \text{for}\ \varkappa<(u-u_1)/u_M.$$

It is also easy to write down the Green's function for the diffusion of thermal neutrons. Using the recipe from Sec. 4, we quickly obtain

$$G_{\mathrm{dif}}(\vec{1},\vec{0})$$

$$=\sum_{m=0}^{\infty}\sum_{a,b=m}^{\infty}\frac{\varepsilon_m(2a+1)(2b+1)}{2(2\pi)^4}\int d^3\mathbf{k}\exp[-i\mathbf{k}(\mathbf{r}_1-\mathbf{r})]$$

$$\times\frac{i}{lk}\Phi_{ab}^m(k,l)P_a^m(\mathbf{k}_0\cdot\Omega)P_b^{-m}(\mathbf{k}_0\cdot\Omega_1)\frac{\cos m(\varphi-\varphi_1)}{\sqrt{A_a^mA_b^m}},$$

$$\Phi_{ab}^m(k,l)=\left(\|AD^m\|/\left(1-\frac{4\pi ih}{lk}\|AD^m\|\right)\right)_{ab}.\qquad(32)$$

Suppose that in an infinite homogeneous medium there are sources $S(\mathbf{r},\Omega,u)$. Then the solution of the kinetic equation for slowing down is written in the form

$$\Psi(\mathbf{r},\Omega,u)=\sum_{m=0}^{\infty}\sum_{a,b=m}^{\infty}\frac{\varepsilon_m(2a+1)(2b+1)}{4\pi}\int_{c-i\infty}^{c+i\infty}\int d^3\mathbf{k}$$

$$\times\frac{\exp(pu+i\mathbf{k}\mathbf{r})}{\sqrt{A_a^m(p)}\,lk}\Phi_{ab}^m(p,k,l)P_a^m(\mathbf{k}_0\cdot\Omega)S_b^m(p,\mathbf{k},\varphi);$$

$$S_b^m(p,\mathbf{k},\varphi)=\frac{1}{2\pi^4}\int_0^\infty du_1\int d^3\mathbf{r}_1\int d^2\Omega_1 S(\mathbf{r}_1,\Omega_1,u_1)$$

$$\times\frac{\exp[-pu_1-i\mathbf{k}\mathbf{r}_1]}{\sqrt{A_b^m(p)}}P_b^{-m}(\mathbf{k}_0\cdot\Omega_1)\cos m(\varphi-\varphi_1).\qquad(33)$$

In the particular case of a point source which is isotropic and monochromatic and is located at the point $\mathbf{r}_u$,

$$S=\delta(\mathbf{r}-\mathbf{r}_u)\,\delta(u),$$

by expanding $\exp i\mathbf{k}(\mathbf{r}-\mathbf{r}_u)$ in a series of products of Legendre polynomials and Bessel functions (cf. reference 3) and integrating over the surface of a sphere of radius k we get:

$$\Psi(\mathbf{r},\Omega,u)=\sum_{a=0}^{\infty}\frac{(2a+1)i}{2\pi^2l}$$

$$\times L^{-1}(p,u)\int_0^\infty\sqrt{k}\,dk\left[\frac{\pi}{|\mathbf{r}_u-\mathbf{r}|A_a(p)A_0(p)}\right]^{1/2}\qquad(34)$$

$$\times\Phi_{a0}(p,k,l)P_a\left(\Omega\frac{\mathbf{r}_u-\mathbf{r}}{|\mathbf{r}_u-\mathbf{r}|}\right)J_{a+1/2}(k|\mathbf{r}_u-\mathbf{r}|).$$

The arguments of the Legendre polynomials and Bessel functions correspond to the fact that Ψ depends only on the angle between the momentum and the radius vector drawn from the source, and on the magnitude of this vector. The term $a=0$ gives the part of Ψ that does not depend on the direction of the momentum:

$$\Psi_0=\frac{i\sqrt{2}}{2\pi^2l}L^{-1}(p,u)\int_0^\infty dk\frac{\sin k|\mathbf{r}_u-\mathbf{r}|}{|\mathbf{r}_u-\mathbf{r}|A_0(p)}\Phi_{00}(p,k,l).\qquad(35)$$

6. EXAMINATION OF THE SOLUTION AND SPECIAL CASES

The function Ψ_0 given by Eq. (35) is a sum in which the κ-th term is of the form

$$\Psi_{0\varkappa} = \frac{iV\overline{2}}{2\pi^2 l} L^{-1}(p, u) \int_0^\infty \frac{\sin k\,|\mathbf{r}-\mathbf{r}_u|}{A_0(p)\,|\mathbf{r}_u-\mathbf{r}|} \left(\frac{4\pi i}{lk}\right)^\varkappa (\|AD\|^{\varkappa+1})_{00}\, dk = \frac{iV\overline{2}}{2\pi^2 l} L^{-1}(p) \int_0^\infty \frac{\sin k\,|\mathbf{r}-\mathbf{r}_u|}{A_0(p)\,|\mathbf{r}_u-\mathbf{r}|} \left(\frac{4\pi i}{lk}\right)^\varkappa \Big\{ \left[A_0(p)\, Q_0\!\left(\frac{i}{lk}\right)\right]^{\varkappa+1}$$

$$+ \varkappa [A_0(p)]^\varkappa A_1(p) \left[Q_0\!\left(\frac{i}{kl}\right)\right]^{\varkappa-1} \left[Q_1\!\left(\frac{i}{kl}\right)\right]^2 + \frac{(\varkappa-2)(\varkappa-3)}{2} [A_0(p)]^{\varkappa-2} [A_1(p)]^2 \qquad (36)$$

$$\times \left[Q_0\!\left(\frac{i}{kl}\right)\right]^{\varkappa-3} \left[Q_1\!\left(\frac{i}{kl}\right)\right]^4 + \varkappa [A_0(p)]^{\varkappa+1} A_2(p) \left[Q_0\!\left(\frac{i}{kl}\right)\right]^{\varkappa-1} \left[Q_2\!\left(\frac{i}{kl}\right)\right]^2 + \ldots \Big\}\, dk.$$

Here we have written out several terms of the matrix element, arranged in the order of decreasing size near $k = 0$. The integrands can easily be summed:

$$\Psi_0 = \sum_{\varkappa=0}^{\infty} \Psi_{0\varkappa} = \frac{iV\overline{2}}{2\pi^2 l} L^{-1}(p) \int_0^\infty \frac{\sin k\,|\mathbf{r}_u-\mathbf{r}|\,dk}{|\mathbf{r}_u-\mathbf{r}|} \Big\{ \frac{Q_0(i/lk)}{1-(4\pi i/lk)\,A_0(p)\,Q_0(i/lk)} - \frac{4\pi i}{lk} \frac{A_1(p)\,[Q_1(i/lk)]^2}{[1-(4\pi i/lk)\,A_0(p)\,Q_0(i/kl)]^2}$$

$$+ \left(\frac{4\pi i}{lk}\right)^2 \frac{A_0(p)\,[A_1(p)]^2\,[Q_1(i/lk)]^4}{[1-(4\pi i/lk)\,A_0(p)\,Q_0(i/kl)]^3} - \frac{4\pi i}{lk} \frac{A_2(p)\,[Q_2(i/kl)]^2}{[1-(4\pi i/lk)\,A_0(p)\,Q_0(i/lk)]^2} + \ldots \Big\}. \qquad (37)$$

The first term in Eq. (37) is the only one in the case of scattering that is symmetrical in the laboratory reference system. It agrees with the solution of Placzek and Volkoff given in reference 1 (single-velocity approximation), as one verifies without difficulty by carrying out the integration in the complex plane. The result consists of the residues at singularities given by the equation

$$1 - \frac{4\pi i}{lk} A_0(p) \frac{\tan^{-1} kl}{kl} = 0$$

and the integral around the cut $(i/l, i\infty)$. This procedure will be carried out below in general form for the partial probabilities. If we suppose that only $A_0(p)$ and $A_1(p)$ are different from zero, then all the terms containing $A_1(p)$ can also be summed easily. It can be seen from Eqs. (36) and (37), however, that the term containing $[A_1(p)]^2$ is already of the same order as the term in $A_2(p)$.

The procedure of operating with matrices in the partial probabilities makes it possible to avoid cumbersome calculations with infinite determinants, such as occur, for example, in Wick's method (cf. references 1 and 4).

It is preferable, however, to use not the expressions (37), which are relatively complicated, but the partial probabilities (36), after first transformed them to a more convenient form. For this purpose we convert the integral over k into a complex Fourier integral. It is easy to do this if we note that

$$(\|AD(i/kl)\|^{\varkappa+1})_{00} = (-1)^{\varkappa+1} \|AD(-i/kl)\|_{00}^{\varkappa+1},$$

in view of the fact that

$$P_a(i/kl)\,Q_b(i/kl) = -P_a(-i/kl)\,Q_b(-i/kl)\,e^{-(a-b)\pi i}$$

and the circumstance that the matrix elements contain sums of terms of the form $D_{0a}D_{ab}D_{bc} \ldots D_{h0}$.

After this, noting that the points $k = \pm i/l$ are branch points and there are no other singularities, we transform the integral along the real axis into an integral along the cut $(i/l, i\infty)$ and introduce the new variable $x = i/kl$. The result is

$$\Psi_{0\varkappa} = \frac{iV\overline{2}}{4\pi^2 l} L^{-1}(p, u) \int_0^1 dx \frac{(4\pi x)^\varkappa \exp(-|\mathbf{r}_u-\mathbf{r}|/lx)}{x^2 A_0(p)\,|\mathbf{r}_u-\mathbf{r}|}$$

$$\times \{\|AD(x-i0)\|^{\varkappa+1} - \|AD(x+i0)\|^{\varkappa+1}\}_{00} = \frac{V\overline{2}}{2\pi^2 l} \int_0^1 dx$$

$$\times \frac{\exp(-|\mathbf{r}_u-\mathbf{r}|/lx)\,(4\pi x)^\varkappa}{x^2\,|\mathbf{r}_u-\mathbf{r}|} \Big\{ L^{-1}(p, u)\,[A_0(p)]^\varkappa\,[B(x)]^{\varkappa+1}$$

$$\times U_{\varkappa+1}\!\left(\frac{Q_0(x)}{B(x)}\right) \qquad (38)$$

$$+ L^{-1}(p, u)\,[A_0(p)]^{\varkappa-1} A_1(p)\,[B(x)]^{\varkappa-1} \Big[x^2 [B(x)]^2$$

$$\times U_{\varkappa+1}\!\left(\frac{Q_0(x)}{B(x)}\right) - 2xB(x)\,U_\varkappa\!\left(\frac{Q_0(x)}{B(x)}\right)$$

$$+ U_{\varkappa-1}\!\left(\frac{Q_0(x)}{B(x)}\right)\Big] + \ldots \Big\},$$

where $U_\kappa(x) = \sin(\kappa \cos^{-1} x)$ are Chebyshev polynomials of the second kind and

$$D_{ab}(x \pm i0) = P_a(x)\left[Q_b \mp \frac{\pi i}{2} P_b(x)\right] \text{ for } a \leqslant b;$$

$$B(x) = \sqrt{Q_0^2(x) + \pi^2/4}.$$

It is easily shown that the integrand has a sharp maximum at $x = 1$. The maximum is due not only to the presence of the exponential, but also to the logarithmic singularity at this point, $B(1) = \infty$. It is easy to see that the first term dominates the others, since near $x = 1$ it is larger than the others by at least a factor $(\ln \infty)^2$.

As for the inverse transformation of the coefficients that depend on p, this is easy to do by using the appropriate formulas given in reference 5. For example, in the case of a single element

$$A_0(u,\varkappa) = L^{-1}(p,u)\,[A_0(p)]^{\varkappa} = \frac{(M+1)^{2\varkappa}e^{-u}}{(16\pi M)^{\varkappa}(\varkappa-1)!} \times \sum_{n=0}^{\varkappa}\binom{\varkappa}{n}(-1)^n(u-nu_M)^{\varkappa-1}\gamma(u-nu_M);$$

$$L^{-1}(p,u)\,[A_0(p)]^{\varkappa-1}A_1(p) = \int\limits_{u-u_M}^{u} A_0(u_1,\varkappa)\,A_1(u-u_1)\,du_1.$$

The first of these formulas agrees with the well known relation (cf. reference 1) for the energy distribution after a given number of collisions. Marshak's review article also gives the extension to the case of a mixture of elements, and asymptotic values of the function $A_0(u, x)$, which has a sharp maximum at $\kappa = u/2u_M$. It is not hard to show by the method of mathematical induction that for $\kappa < u/u_M$, i.e., when the summation in Eq. (39) is not broken off, $A_0(u, \kappa) \equiv 0$; this is physically obvious.

Thus Eqs. (37) or (38) and (39) give expressions in quadratures for Ψ_0 and $\Psi_{0\kappa}$. It is clear that for the value of κ equal to the average number of collisions needed for slowing down to a given energy one will find a maximum of $\Psi_{0\kappa}$. Therefore it is sufficient to include a small number of the Ψ_0 which are in the neighborhood of this average κ. Owing to the rapid decrease of the integrand (particularly for large κ) the integration does not involve much labor.

From the structure of the formula (36) one can see how the results are to be extended to the case of a nonconstant mean free path. In first approximation it is obvious that we are to make the replacement

$$\left(AD\left(\frac{i}{kl}\right)\|^{\varkappa+1}\right)_{00} \to \left(\prod_{\alpha=0}^{\varkappa}\left\|AD\left(i/k\left[l(0)+\alpha\frac{l(u)-l(0)}{\varkappa}\right]\right)\right\|\right)_{00}.$$

7. COMPARISON OF EXISTING METHODS; SUMMARY

In methods existing hitherto simplifying approximations have been introduced directly into the equation. Wick's method[1,4] is based on expanding the functions f and Ψ in series and obtaining an infinite system of equations. One then determines approximately the zeros of the infinite determinant in the denominator of the integrand. In the one-velocity method[1] and in the considerably better developed method of Temkin[6] only the scattering function is expanded in series (which is broken off after a certain term). In reference 6 the main part of the solution is found, corresponding to spherical symmetry of the scattering, and then a perturbation method is given for finding corrections.

The main shortcomings of these methods lie in the fact that their formalism (1) is not related to the physical peculiarities of the phenomenon, and (2) does not make it possible to write down an exact solution in the form of combinations of known functions and operators.

The advantage of the method of partial probabilities lies in the fact that it introduces explicitly a new physical variable — the number of collisions of a particle with nuclei of the medium. The formalism has a clear physical meaning, connects the entire problem with the theory of probability, and reduces it to a problem of multiple integration. The method deals with the actual physical picture of successive transfers of neutrons or γ-ray quanta. It makes it possible to write down in a compact form the exact solution of the kinetic equation as a function of all six variables, and then, by the use of the available apparatus of multiple integration, to bring it into a simple form convenient for calculation.

The entire method set forth above is in principle capable of being applied also to boundary-value problems. For this purpose it is necessary to choose the zeroth-order partial probability in such a way that it satisfies the boundary conditions.

APPENDIX

We shall show that

$$D_{ab}^m(y) = \frac{1}{2}\int\limits_{-1}^{+1}\frac{P_a^m(x)\,P_b^m(x)}{y-x}\,dx = \begin{cases} P_a^m(x)\,Q_b^m(x) & a \leqslant b \\ Q_a^m(x)\,P_b^m(x) & a \geqslant b. \end{cases} \qquad \text{(A.1)}$$

where y is not on the segment $(-1, +1)$ and

$$Q_a^m(y) = (y^2-1)^{m/2}d^mQ_a(y)/dy^m$$

are associated Legendre functions of the second kind (cf. reference 7).

To obtain the proof we first calculate

$$D_{ma}^m(y) = \frac{1}{2}\int\limits_{-1}^{+1}\frac{P_a^m(x)\,P_m^m(x)}{y-x}\,dx. \qquad \text{(A.2)}$$

Integrating by parts and differentiating with respect to y, we get:

$$\frac{d}{dy}D_{ma}^m(y) = \frac{2m-1}{2}(a+m)(a-m+1)\,D_{m-1,a}^{m-1}(y). \qquad \text{(A.3)}$$

We use the relation

$$\frac{d}{dx}[(1-x^2)^{m/2}P_a^m(x)] = (2m-1)(a+m)(a-m+1)(1-x^2)^{(m-1)/2}P_a^{m-1}(x),$$

and the obvious fact that for any value of m

$$D_{ma}^{m}(\infty)=0. \qquad (A.4)$$

Equation (A.3) at once gives

$$\frac{d^m}{dy^m}D_{ma}^{m}(y)=\frac{(2m-1)!!}{2}\frac{(a+m)!}{(a-m)!}D_{0a}^{0}$$

$$=(2m-1)!!\,\frac{(a+m)!}{(a-m)!}Q_a(y). \qquad (A.5)$$

Integrating Eq. (A.5) with the supplementary condition (A.4), we get

$$D_{ma}^{m}(y)=(2m-1)!!\,\frac{(a+m)!}{(a-m)!}\int_y^\infty\int_y^\infty\ldots\int_y^\infty Q_a(y)\,d^m y$$

$$=(2m-1)!!\,\frac{(a+m)!}{(a-m)!}Q_a^{-m}(y)(y^2-1)^{m/2}=P_m^m(y)\,Q_a^m(y).$$

We next take the case, with a $\geq m+1$:

$$D_{m+1,a}^{m}(y)=\frac{1}{2}\int_{-1}^{+1}\frac{P_{m+1}^m(x)\,P_a^m(x)}{y-x}dx$$

$$=\frac{2m+1}{2}\int_{-1}^{+1}\frac{xP_m^m(x)\,P_a^m(x)}{y-x}dx$$

$$=\frac{2m+1}{2}y\int_{-1}^{+1}\frac{P_m^m(x)\,P_a^m(x)}{y-x}dx=P_{m+1}^m(y)\,Q_a^m(y).$$

Here we have used the orthogonality of the associated Legendre polynomials. Thus the theorem (A.1) is proved for b = m and b = m + 1. It is now easy to prove it for arbitrary a and b by the method of mathematical induction, by using the recurrence relation

$$P_{a+1}^m(x)=\frac{2a+1}{a-m+1}xP_a^m(x)-\frac{a+m}{a-m+1}P_{a-1}^m(x).$$

[1] R. Marshak, Revs. Mod. Phys. **19**, 185 (1947).

[2] S. Glasstone and M. Edlund, The Elements of Nuclear Reactor Theory, Van Nostrand, 1952.

[3] G. N. Watson, A Treatise on the Theory of Bessel Functions, Chapter 4, Cambridge, 1945.

[4] G. Wick, Phys. Rev. **75**, 738 (1949).

[5] Ditkin and P. Kuznetsov, Справочник по операционному исчислению (Handbook of Operational Calculus), GITTL 1951.

[6] A. S. Temkin, Прикладная геофизика (Applied Geophysics) 17, Gostoptekhizdat 1957.

[7] E. W. Hobson, Theory of Spherical and Ellipsoidal Harmonics.

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