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*VELOCITY AND TEMPERATURE DISCONTINUITIES NEAR THE WALLS OF A BODY AROUND WHICH RAREFIED GASES FLOW WITH TRANSONIC VELOCITIES**

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New and more general formulas are derived for velocity and temperature discontinuities on a gas-wall surface for rarefied gas flows of arbitrary Mach number. The equation for the velocity discontinuity is practically the same as that for $M \ll 1$; on the other hand, the relation for the temperature discontinuity differs markedly from the well-known Maxwell formula for a gas at rest near a wall.

In previous investigations, even in the most detailed,[1] the effects of the slipping of rarefied gases along the walls and the temperature discontinuities on the gas-wall boundary have been studied only in cases corresponding to $M \ll 1$. Here M is a dimensionless quantity, equal to the ratio of the speed of flow far from the wall to the speed of sound (the Mach number). Furthermore, there is a great need to know the laws governing these effects for flows with arbitrary values of M, since the gas dynamics of rarefied gases are of considerable interest at the present time, principally in connection with the problem of the flight of rocket missiles and apparatus at the upper levels of the atmosphere.

The work below had as its aim the solutions of these problems.

1. INITIAL ASSUMPTIONS; THE VELOCITY DISTRIBUTION FUNCTION f

As is well known,[2] the equations of gas dynamics preserve their usual form in relation to the expression for the heat flow q_μ and the stress tensor $\tau_{\mu\nu}$ if

$$Ml/L = M^2/R < 1, \tag{1.1}$$

where l is the length of the molecular mean free path, L a characteristic linear dimension of the object (or channel) in the flow, and R the Reynolds number. q_μ and $\tau_{\mu\nu}$ are in this case expansions in powers of the parameter Ml/L, with factors in the form of first, second and higher derivatives, with respect to the coordinates x_α, of the mean velocity $\bar{u}_\mu$ and of the temperature T.

We assume condition (1.1). Then

$$\rho\bar{u}_\nu = \rho v_\nu, \qquad \rho\overline{u_\mu u_\nu} = p\delta_{\mu\nu} - \tau_{\mu\nu} + \rho v_\mu v_\nu,$$
$$\rho\left(\frac{1}{2}\overline{u^2 u_\nu} + \varepsilon\overline{u_\nu}\right) = pv_\nu + q_\nu - \tau_{\mu\nu}v_\mu + \rho v_\nu\left(\frac{v^2}{2} + c_v T\right), \tag{1.2}$$

where v_μ is the mean velocity of the macroscopic motion of the medium, ϵ is the internal energy of the molecule of the gas, and c_V is the specific heat at constant volume.

The laws of conservation of mass, momentum, and energy at the gas-wall boundary can be written in the following form, if we denote the unit normal vector by n_ν:

$$\rho\bar{u}_\nu n_\nu = (\rho\bar{u}_\nu)^* n_\nu, \qquad \overline{\rho u_\mu u_\nu} n_\nu = \rho(\overline{u_\mu u_\nu})^* n_\nu,$$
$$\rho\left(\frac{1}{2}\overline{u_\mu u_\mu u_\nu} + \overline{\varepsilon u_\nu}\right)n_\nu = \rho\left(\frac{1}{2}\overline{u_\mu u_\mu u_\nu} + \overline{\varepsilon u}\right)^* n_\nu. \tag{1.3}$$

*This work was completed in 1950.

In these equations, the asterisk denotes quantities produced by the flow of molecules incident on the wall and reflected from it, while the quantities without the asterisk refer to the layer of gas infinitesimally close to the wall.

The velocity distribution function, which is a solution of the Boltzmann equation for the given problem, should have the form

$$f = f_M(1+\psi),\ \psi \ll 1, \tag{1.4}$$

$$f_M = (h/\pi)^{3/2}\exp\{-h(u_\mu - v_\mu)(u_\mu - v_\mu)\}, \tag{1.5}$$

where f_M is the well-known Maxwell distribution function,

$$\psi = A + B(u_\mu - v_\mu)(u_\mu - v_\mu) + C\tau_{\alpha\beta}(u_\alpha - v_\alpha)(u_\beta - v_\beta) + [D + E(u_\alpha - v_\alpha)(u_\alpha - v_\alpha)](u_\beta - v_\beta)q_\beta. \tag{1.6}$$

Furthermore, the conditions of thermodynamic equilibrium should be satisfied for f and ψ at each point of the medium and at each instant of time:

$$\int f\,du_1du_2du_3 = \int f d\omega = \int f_M\,d\omega = 1,$$
$$\int fu_\alpha d\omega = \int f_M u_\alpha d\omega = v_\alpha, \tag{1.7}$$
$$\int \xi_\alpha\xi_\alpha\,d\omega = \int (u_\alpha - v_\alpha)(u_\alpha - v_\alpha)\,d\omega = \int a^2 f d\omega = \int a^2 f_M\,d\omega.$$

The constants A, B, C, D, and E should be chosen to satisfy conditions (1.7) and (1.2). This leads to a system of equations for the constants:

$$\int [A + Ba^2 + C(\tau_{11}\xi_1^2 + \tau_{22}\xi_2^2 + \tau_{33}\xi_3^2)]\,f_M\,d\omega = 0,$$
$$\int \xi_\alpha\xi_\beta f_M(D + Ea^2)\,d\omega = 0, \tag{1.7a}$$

$$\int [A + Ba^2 + C(\tau_{11}\xi_1^2 + \tau_{22}\xi_2^2 + \tau_{33}\xi_3^2)]\,a^2 f_M\,d\omega = 0;$$
$$\rho\int (A + Ba^2 + C\tau_{\alpha\beta}\xi_\alpha\xi_\beta)\,f_M\xi_\mu\xi_\nu\,d\omega = -\tau_{\mu\nu}, \tag{1.2a}$$
$$\frac{\rho}{2}\int (D + Ea^2)\,a^2\xi_\mu^2\,d\omega = 1.$$

In these expressions, the quantities ξ_μ and a^2 are determined by means of (1.7), while we denote by ξ_μ^2 simply the square of the component ξ_μ, and the sum of squares of ξ_α is written, as usual, in the form $\xi_\alpha\xi_\alpha = a^2$.

In order that the quantities A, B, C, . . . be independent of $\tau_{\mu\nu}$ and q_μ, it is necessary to employ the Stokes hypothesis:

$$\tau_{11} + \tau_{22} + \tau_{33} = 0, \tag{1.8}$$

satisfaction of which is implied in the choice of the distribution function in the form (1.6).

Carrying out integration in Eqs. (1.7a) and (1.2a), and solving the resultant set of equations, we find

$$A = B = 0,\quad C = -2h^2/\rho;\quad D = -4h^2/\rho;\quad E = 1.6\,h^3/\rho. \tag{1.9}$$

2. VELOCITY AND TEMPERATURE DISCONTINUITIES ON THE GAS-WALL INTERFACE

We assume that a part s of the molecules is reflected diffusely from the wall with a Maxwellian velocity distribution f'_M, corresponding to a certain effective temperature T′, connected with the wall temperature T_W by the Knudsen accomodation coefficient:

$$T' - T = \alpha(T_w - T). \tag{2.1}$$

Further, we let the remaining faction of the molecules (1 – s) be reflected specularly from the wall. Then the first of equations (1.3) can be written in the form

$$s\Big(\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{-\infty}^{+\infty} u_2 f'_M d\omega + \int_{-\infty}^{+\infty}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} fu_2\,d\omega\Big)$$
$$+(1-s)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} fu_2 d\omega = 0,$$

regarding the normal **n** to be directed along the x_2 axis, inside the gas. Or, taking it into account that the latter integral is equal to zero, we shall have, leaving in what follows, for brevity, one integral with the designation of the limits of integration with respect to the variable x_2:

$$\int_0^\infty u_2 f'_M\,d\omega + \int_{-\infty}^{0} u_2 f\,d\omega = 0. \tag{2.2}$$

Further, in accord with the second equation of (1.3), and considering that

$$\int_0^\infty f'_M u_\alpha u_2\,d\omega = 0 \text{ for } \alpha \neq 2,$$

we obtain

$$\int_0^\infty fu_\alpha u_2\,d\omega - (1-s)\int_0^{-\infty} fu_\alpha u_2\,d\omega = 0, \tag{2.3}$$

$$\int_0^\infty fu_2^2\,d\omega = s\int_0^\infty f'_M u_2^2 d\omega + (1-s)\int_0^{-\infty} fu_2^2\,d\omega. \tag{2.4}$$

Finally, the last equation of (1.3) yields:

$$2[-v_\alpha\tau_{\alpha 2} + q_2] = \rho s\Big\{\int_0^\infty f'_M u_2(u^2 + 2\varepsilon)\,d\omega + \int_{-\infty}^{0} f(u^2 + 2\varepsilon)\,u_2 d\omega\Big\}. \tag{2.5}$$

Equations (2.2) to (2.5) express the boundary conditions for the gas dynamic flow of rarefied gases. For their formulation in concrete form, however, it is necessary to calculate the integrals involved, making use of the distribution function f found above; in this case we assume for molecules reflected with accomodation

$$f'_M = A(1 + b_1 a^2 + b_2 a^4 + b_3 a^6) e^{-h u_\alpha u_\alpha}. \quad (2.6)$$

The coefficients A, b_1, b_2, . . . are determined from the normalization condition $\int f'_M d\omega = 1$ and fulfilment of Eq. (2.2) and two other relations which shall be used later:

$$\int_0^\infty f'_M u_2\left(\frac{u^2}{2} + \varepsilon\right) d\omega = \alpha \int_0^\infty f_M^w u_2 \left(\frac{u^2}{2} + \varepsilon^w\right) d\omega$$

$$-(1-\alpha)\int_{-\infty}^0 f u_2 \left(\frac{u^2}{2} + \varepsilon\right) d\omega, \quad (2.7)$$

$$\int_0^{-\infty} f'_M u_2 d\omega = \alpha \int_0^\infty f_M^w u_2 d\omega - (1-\alpha)\int_{-\infty}^0 f u_2 d\omega. \quad (2.8)$$

Substituting Eq. (1.4) for f in (2.3), we get

$$\int_{-\infty}^{\infty+} f_M (\xi_\alpha \xi_2 + v_\alpha \xi_2 + \psi \xi_\alpha \xi_2 + \psi v_\alpha \xi_2)\, d\omega$$

$$+ s \int_0^{-\infty} f_M (\xi_\alpha \xi_2 + v_\alpha \xi_2 + \psi \xi_\alpha \xi_2 + \psi v_\alpha \xi_2)\, d\omega = 0.$$

Multiplying this relation by ρ and computing the corresponding integrals with the help of (1.5) to (1.9), we get

$$-\tau_{\alpha 2} + \rho v_\alpha \bar{\xi}_2 s + \frac{s}{2}\tau_{\alpha 2} + 0.2\, s \left(\frac{h}{\pi}\right)^{1/2} q_\alpha - \frac{s\, \tau_{22} h^{1/2}}{2\pi^{3/2}} v_\alpha = 0, \quad (2.9)$$

where

$$\bar{\xi}_2 = -\int_{-\infty}^0 \xi_2 f_M d\omega = 1/2\sqrt{h\pi} = \bar{a}/4, \quad (2.10)$$

Here $\bar{a}$ is the mean velocity of the random thermal motion of the molecules entering, according to the molecular kinetic theory, into the expression for the viscosity η.

As is well known,

$$h = 1/2\,(c_p - c_v)\,T = \gamma/2c_p(\gamma - 1)\,T. \quad (2.11)$$

Furthermore, we have for the heat-flow vector and for the tensor of viscous forces

$$q_\alpha = -\lambda \frac{\partial T}{\partial x_\alpha};\ \tau_{\mu\nu} = \eta\left(\frac{\partial u_\mu}{\partial x_\nu} + \frac{\partial u_\nu}{\partial x_\mu} - \frac{2}{3}\delta_{\mu\nu}\frac{\partial u_\alpha}{\partial x_\alpha}\right). \quad (2.12)$$

Finally, we introduce the dimensionless number

$$\mathrm{Pr} = \eta c_p/\lambda. \quad (2.13)$$

Substituting the values of the corresponding quantities λ, q_α, etc. from (2.10) to (2.13) in Eq. (2.9), and taking it into account that in flow over the wall we can always set $\tau_{22} = 0$, we get

$$v_\alpha = \frac{\eta}{\beta}\frac{\partial v_\alpha}{\partial x_2} + \frac{0.2}{(\gamma-1)\,\mathrm{Pr}}\frac{\eta}{\rho T}\frac{\partial T}{\partial x_\alpha}, \quad (2.14)$$

where

$$\eta/\beta = (2/s - 1)(2\eta/\rho\bar{a}). \quad (2.15)$$

In these relations, β is the coefficient of external friction and η/β is the slipping coefficient.

There are several theoretical formulas for the viscosity coefficient. Apparently, the best of these is the Chapman formula:[2]

$$\eta = 0.499\,\rho\bar{a}l. \quad (2.16)$$

Substitution of this value of η in (2.15) gives

$$\eta/\beta = 0.998\,(2/s - 1)\,l. \quad (2.17)$$

According to experimental data, $s \approx 0.8$ to 1 for various surfaces and gases; therefore, $h/\beta \approx l$.

Equations (2.14) to (2.17) for the velocity discontinuity do not contain anything new in comparison with the expressions obtained earlier in molecular-kinetic theory. Their derivation, given above, only furnishes a basis for the possibility of using the ordinary formulas for the velocity discontinuity at a wall at arbitrary values of the Mach number M.

The formulas obtained for the temperature discontinuity are essentially new, however. With the help of Eq. (2.8), we can transform Eq. (2.2) to the form:

$$\int_0^\infty f_M^w u_2\, d\omega + \int_{-\infty}^0 f u_2\, d\omega = 0.$$

Then, in accord with Eq. (2.10),

$$\int_0^\infty f_M^w u_2\, d\omega = -\int_{-\infty}^0 f u_2\, d\omega = -\int_{-\infty}^0 f_M u_2\, d\omega = \frac{\bar{a}}{4}. \quad (2.18)$$

From (2.5) and (2.6) we obtain

$$2(q_2 - v_a \tau_{\alpha 2}) = \rho s \alpha \left\{\int_0^\infty f_M^w u_2 (u^2 + 2\varepsilon^w)\, d\omega + \int_{-\infty}^0 f u_2 (u^2 + \varepsilon)\, d\omega\right\}.$$

In correspondence with Eq. (1.4), we can transform this equation, considering that $v_2 = 0$, to the form

$$q_2 - v_\alpha \tau_{\alpha 2} = {}^1/_2\, s\alpha\rho\,(M_1 + M_2 + M_3), \qquad (2.19)$$

where

$$M_1 = \int_0^\infty f_{\mathrm{M}}^w u_2\,(u^2 + 2\varepsilon^w)\,d\omega, \qquad (2.20)$$

$$M_2 = \int_{-\infty}^0 f_{\mathrm{M}} u_2\,(u^2 + 2\varepsilon)\,d\omega, \qquad (2.21)$$

$$M_3 = \int_{-\infty}^0 f_{\mathrm{M}}\psi\,\{\xi_\alpha\xi_\alpha\xi_2 + 2\xi_\alpha\xi_2 v_\alpha + v_\alpha v_\alpha \xi_2\}\,d\omega. \qquad (2.22)$$

Calculation of these integrals, making use of (2.11) and (2.18), gives

$$M_1 = \bar{a}\,\{c_p T\,(\gamma - 1)/\gamma + \varepsilon^w/2\}, \qquad (2.20a)$$

$$M_2 = -\,\bar{a}\,\{c_p T\,(\gamma - 1)/\gamma + \varepsilon/2 + v^2/4), \qquad (2.22a)$$

$$M_3 = \frac{1}{\rho}\left(q_2 - v_\alpha\tau_{\alpha 2} - 0.8\,\frac{q_\alpha v_\alpha}{\pi\bar{a}}\right)$$

$$+ \tau_{22} C \int_{-\infty}^0 (\xi_2^3 - \xi_3^2\xi_2)\,(1 + a^2)\,f_{\mathrm{M}}\,d\omega. \qquad (2.21a)$$

Substituting the values of M in (2.19) and taking $\tau_{22} = 0$ as before, we get

$$(q_2 - v_\nu\tau_{\nu 2})\left(1 - \frac{s\alpha}{2}\right) + \frac{0.4\,s\alpha}{\pi}\,q_\nu\,\frac{v_\nu}{\bar{a}}$$

$$= \frac{s\alpha\rho\bar{a}}{4}\left\{\frac{2c_p\,(\gamma - 1)}{\gamma}\,(T_w - T) + (\varepsilon^w - \varepsilon) - \frac{v^2}{2}\right\}.$$

From molecular-kinetic theory,

$$\varepsilon^w - \varepsilon = \left(c_v - \frac{3}{2}\,c_p\,\frac{\gamma - 1}{\gamma}\right)(T_w - T) = -\,\Delta T c_p\,\frac{5 - 3\gamma}{2\gamma} \qquad (2.23)$$

With the help of this relation, the preceding equation is transformed to the form

$$(-\,v_\alpha\tau_{\alpha 2} + q_2)\left(\frac{2}{s\alpha} - 1\right) + \frac{0.8}{\pi}\,\frac{q_\alpha v_\alpha}{\bar{a}} = -\,\frac{s\bar{a}}{4}\,\mu c_p\left(\Delta T + \frac{v^2}{c_p\mu}\right), \qquad (2.24)$$

where

$$\mu = (\gamma + 1)/\gamma. \qquad (2.25)$$

Equation (2.24) determines the value of the temperature discontinuity between the gas and the wall for flow past it at Mach numbers $M \gtrless 1$. In contrast with the formula for the velocity discontinuity (2.14), it differs essentially from the well-known Maxwell formula, which corresponds to the case of gas at rest near a wall ($M = 0$) and which can be obtained by putting $v_\alpha = 0$ in (2.24).

In the simpler case of two-dimensional flow in the x_1, x_2 plane (which will occur, for example, for a plane Prandtl boundary layer), we can transform Eq. (2.24) [by making use of Eqs. (2.12) and (2.13)] to the form

$$\Delta T + \frac{v^2}{\mu c_p} = g\,\frac{\partial}{\partial x_2}\left(T + \mathrm{Pr}\,\frac{v_1^2}{c_p}\right) + g_1\,\frac{\partial T}{\partial x_1}, \qquad (2.26)$$

where

$$g = (2/s\alpha - 1)\,(4\eta/\mathrm{Pr}\,\mu\bar{a}s), \qquad (2.27)$$

$$g_1 = (3.2\,\eta/\pi\mu\rho\bar{a}\,\mathrm{Pr})\,(v_1/a), \qquad (2.28)$$

or, if we use the Chapman formula (2.16) for the viscosity,

$$g = (2/s\alpha - 1)\,(1.996\,l/\mu\,\mathrm{Pr}), \qquad (2.27a)$$

$$g_1 = 0.359\,v_1 l/\pi\mu\,\mathrm{Pr}\,\bar{a}. \qquad (2.28a)$$

For computational purposes, it is useful to note that with use of boundary-layer theory, we can neglect in the formulas for velocity and temperature discontinuities, (2.14) and (2.16), the terms containing the derivative $\partial T/\partial x$, if x_1 is directed along the flow in the immediate vicinity of the wall. In addition, we can, with accuracy sufficient for practical purposes, consider $\mu \approx 2$ and $\mathrm{Pr} = 1$. Then Eq. (2.26) takes the form

$$\Delta\Theta = g\partial\Theta/\partial x_2, \qquad (2.26b)$$

which coincides with the known Maxwell formula if we replace the ordinary temperature T in it by the "throttling" temperature used in gas dynamics:

$$\Theta = T + v^2/2c_p. \qquad (2.29)$$

Once more, it should be noted that by setting $v_1 = 0$ and $s = 1$, (2.26) and (2.27) reduce to the known Maxwell expressions for the temperature discontinuity on the gas-wall interface [if the gas is at rest relative to the wall ($M = 0$)].

In conclusion, we point out that instead of (2.26) and (2.14) one can easily obtain if necessary more general formulas for velocity and temperature discontinuities by substituting in Eqs. (2.9) and (2.24) the expressions for heat flow and the stress tensor,[2] taking into account terms of higher order in M^2/R with higher derivatives. The relations obtained by such a method are, however, too cumbersome to cite here.

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