INTERACTION BETWEEN ELECTRONS AND LATTICE VIBRATIONS IN A NORMAL METAL

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A method is developed which enables one to obtain the electron-energy spectrum and dispersion of the lattice vibrations without assuming that the interaction between electrons and phonons is small.

1. INTRODUCTION

THE attraction between electrons due to the exchange of phonons leads in superconductors to the formation of a bound state of two electrons with opposite momenta. In the ground state of a superconductor a condensed component consisting of these bound electrons is formed and a gap results in the energy spectrum.[1]

In papers on the theory of superconductivity[1] the interaction between electrons and lattice vibrations has been assumed to be small, although we know that this condition is not fulfilled for all superconductors. It is therefore of interest to construct a theory which is not limited in this way.

In the present paper we develop a method which enables one to consider the interaction between electrons and lattice vibrations in a normal metal without assuming that the interaction is small. The method is based on the use of quantum field-theoretical equations.

The application of field theory to superconductors involves certain difficulties. The state which contains the "condensate" of bound electrons cannot be obtained from the ground state of noninteracting particles by applying the interaction adiabatically. The necessary condition for the use of ordinary field-theoretical methods is thus violated. The method developed below for a normal metal, where this difficulty does not occur, can therefore be extended to a superconductor only through a separate investigation.

The interaction between electrons and lattice vibrations in a normal metal is certainly of interest in itself. Fröhlich[2] used perturbation theory to investigate this interaction. He considered an isotropic model of a metal described by the Hamiltonian

$$H = H_0 + H_1,\ H_0 = \sum_{\mathbf{p}} \varepsilon_{\mathbf{p}}^0 a_{\mathbf{p}}^+ a_{\mathbf{p}} + \sum_{q<q_m} \omega_{\mathbf{q}}^0 b_{\mathbf{q}}^+ b_{\mathbf{q}},$$
$$H_1 = \sum_{\mathbf{p},\, q<q_m} \alpha_{\mathbf{q}} a_{\mathbf{p}+\mathbf{q}}^+ a_{\mathbf{p}} (b_{\mathbf{q}} + b_{-\mathbf{q}}^+), \qquad (1)$$

where $a_{\mathbf{p}}$, $a_{\mathbf{p}}^+$ and $b_{\mathbf{q}}$, $b_{\mathbf{q}}^+$ are the annihilation and creation operators of electrons and phonons and q_m is the maximum phonon momentum. We know that $\alpha_{\mathbf{q}}^2$, which determines the interaction between electrons and phonons, is given for small q (in atomic units) by

$$\alpha_q^2 = (\lambda_0 \pi^2/p_0)\, \omega_{\mathbf{q}}^0,\ \omega_{\mathbf{q}}^0 = c_0 q, \qquad (2)$$

where c_0 is the unrenormalized velocity of sound, $c_0 \sim M^{-1/2}$, M is the mass of an ion and λ_0 is a dimensionless parameter, introduced by Fröhlich,[2] which does not contain the ion mass; $\lambda_0 \lesssim 1$.

It will be shown below that the energy spectrum of the Hamiltonian (1) cannot be obtained by perturbation theory, despite the smallness of the parameter $M^{-1/2}$ in $\alpha_{\mathbf{q}}^2$. The criterion for the applicability of perturbation theory is the smallness of λ_0, which does not contain the ion mass. Field theoretical methods[3] enable us to obtain the energy spectrum, without assuming that λ_0 is small, as a power series in $M^{-1/2}$.

2. METHOD OF SOLUTION

We introduce the electron and phonon propagation functions G and D:

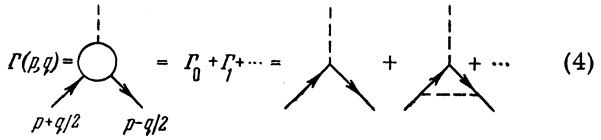
$$G = i \langle T\Psi(1)\, \Psi^+(2)\rangle,\ D = i \langle T\varphi(1)\, \varphi(2)\rangle, \qquad (3)$$

where the averaging is performed over the ground state of the system

$$\varphi = e^{iHt} \sum_{q<q_m} (b_{\mathbf{q}} + b_{-\mathbf{q}}^+)\, e^{i\mathbf{q}\mathbf{r}} \alpha_{\mathbf{q}} e^{-iHt},$$
$$\Psi = e^{iHt} \sum_{\mathbf{p}} a_{\mathbf{p}} e^{i\mathbf{p}\mathbf{r}} e^{-iHt}.$$

Dyson's equations relate D and G to the vertex

part Γ, which is defined by the following set of diagrams:



(4)

Here $q = (\mathbf{q}, \omega)$ and $p = (\mathbf{p}, \epsilon)$. The interaction energy in (1) can be written in the form

$$H_1 = \sum_{\substack{p \\ q<q_m}} \Psi^+_{p+q}\Psi_p\varphi_q,$$

where $\Psi_{\mathbf{p}}$ and $\varphi_{\mathbf{q}}$ are spatial Fourier components of the operators in the Green's functions. Therefore the first diagram in (4) corresponds to $\Gamma = \Gamma_0 = 1$. It will be shown below that the following terms in (4) are of the order of $M^{-1/2}$. Therefore Γ can be replaced by 1 in Dyson's equations, after which a closed system of equations is obtained for D and G. In the momentum representation Dyson's equations are

$$\begin{aligned} G(p) &= G_0(p) + G_0(p)\,\Sigma(p)\,G(p), \\ \Sigma(p) &= \frac{1}{i}\int G(p-q)\,D(q)\,\Gamma\left(p-\frac{q}{2}, q\right)d^4q, \\ D(q) &= D_0(q) + D_0(q)\,\Pi(q)\,D(q), \\ \Pi(q) &= \frac{1}{i}\int G\left(p+\frac{q}{2}\right)G\left(p-\frac{q}{2}\right)\Gamma(p,q)\,d^4p, \end{aligned} \tag{5}$$

where

$$d^4p = d\mathbf{p}\,d\varepsilon/(2\pi)^4, \quad d^4q = d\mathbf{q}\,d\omega/(2\pi)^4,$$

Σ and Π are the irreducible parts of the electron and phonon self energies, and D_0 and G_0 are the electron and phonon Green's functions in the absence of interaction:[2]

$$\begin{aligned} G_0(p) &= \frac{1}{\varepsilon^0_{\mathbf{p}} - \varepsilon - i\Delta(p)}, \\ D_0(q) &= \alpha_q^2\left\{\frac{1}{\omega_q^0 - \omega - i\delta} + \frac{1}{\omega_q^0 + \omega - i\delta}\right\}, \end{aligned} \tag{6}$$

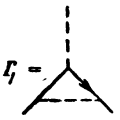
where

$$\Delta(p) \to \begin{cases} +0 & p > p_0, \\ -0 & p < p_0 \end{cases}, \quad \delta \to +0.$$

Assuming $\Gamma = 1$ in (5), we obtain G and D. As was shown in reference 3, the energy spectrum is determined by the poles of the analytic continuation of $G(\mathbf{p}, \epsilon)$ and $D(\mathbf{q}, \omega)$ in the complex plane.

3. THE VERTEX PART

We shall show that the vertex part differs from $\Gamma_0 = 1$ by a quantity of the order of $M^{-1/2}$. Let us consider a first-order perturbation correction to Γ.



We shall assume that

$$p \sim p_0, \quad \varepsilon \sim \mu_0, \quad \omega \lesssim \omega_q|_{q=2p_0} = \omega_0,$$

since for our further calculations this is the important range of values of p and ω. From our definition of G and D each internal line of the diagrams corresponds to a Green's function divided by i. We obtain

$$\Gamma_1(p, q) = i\int D_0(p - p_1)\,G_0\left(p_1 + \frac{q}{2}\right)G_0\left(p_1 - \frac{q}{2}\right)dp_1. \tag{7}$$

In accordance with (6), the function $D_0(\mathbf{p} - \mathbf{p}_1, \epsilon - \epsilon_1)$ possesses the following properties:

When $|\mathbf{p} - \mathbf{p}_1| \sim p_0$ and $\epsilon - \epsilon_1 \ll \omega_0$, D_0 can be replaced by

$$D_0 = 2\pi^2\lambda_0/p_0.$$

When $\epsilon - \epsilon_1 \gg \omega_0$, D_0 diminishes as $(\epsilon - \epsilon_1)^{-2}$. For $|\mathbf{p}_1 - \mathbf{p}| > q_m$, $D_0 = 0$. Using these properties of D_0, we obtain for Γ_1:

$$\Gamma_1 \sim \frac{\lambda_0 i}{8\pi^2 p_0}\int_{\varepsilon-\omega_0/2}^{\varepsilon+\omega_0/2} d\varepsilon_1 \int_{|\mathbf{p}-\mathbf{p}_1|<q_m} G_0\left(\mathbf{p}_1 + \frac{\mathbf{q}}{2}, \varepsilon_1 + \frac{\omega}{2}\right) \times G_0\left(\mathbf{p}_1 - \frac{\mathbf{q}}{2}, \varepsilon_1 - \frac{\omega}{2}\right)d\mathbf{p}_1. \tag{8}$$

Integration with respect to ϵ_1 gives the factor $\omega_0 \sim M^{-1/2}$ and leads to

$$\Gamma_1 \sim \lambda_0\omega_0/p_0^2 \sim \lambda_0/\sqrt{M},$$

if integration over $\mathbf{p}_1$ does not introduce factors $\sim 1/\omega_0$. Such factors result only for small $q \lesssim \omega_0/p_0$ and $\omega < \omega_0$, when the two poles of the integrand approach each other. Then the integrand has a maximum near $\mathbf{p}_1 = \mathbf{g}$, where g is given by $\epsilon^0_{\mathbf{g}} = \epsilon_1 \simeq \epsilon$ and thus $g \sim p_0$. Integration over regions far from this maximum does not introduce factors $\sim 1/\omega_0$. We can therefore limit ourselves to consideration of the integral over $\mathbf{p}_1$ in the region $(\mathbf{p}_1 - \mathbf{g})/g \ll 1$. Using (6) and the notation $\epsilon^0_{\mathbf{p}_1} - \epsilon = E$, we obtain from (8)

$$\Gamma_1 \sim i\lambda_0 p_0 \int_{\varepsilon-\omega_0/2}^{\varepsilon+\omega_0/2} d\varepsilon_1 \int_{-1}^{1} dx \int_{-\infty}^{\infty} dE \Big/ \left[E + \frac{v_g q x}{2} - \frac{\omega}{2} - i\Delta\left(\varepsilon_1 + \frac{\omega}{2}\right)\right]\left[E - \frac{v_g q x}{2} + \frac{\omega}{2} - i\Delta\left(\varepsilon_1 - \frac{\omega}{2}\right)\right].$$

For simplicity it is assumed here that $q_m > p + g$, and the condition $|\mathbf{p}-\mathbf{p}_1| < q_m$ imposes no limitation on integration near $p_1 = g$.

Integration with respect to E gives

$$\Gamma_1 \sim \lambda_0 p_0 \int_{\varepsilon-\omega_0/2}^{\varepsilon+\omega_0/2} d\varepsilon_1 \int_{-1}^{1} dx \frac{\theta(\varepsilon_1-\mu_0+\omega/2)-\theta(\varepsilon_1-\mu_0-\omega/2)}{v_g q x-\omega+i\delta\omega/|\omega|},$$

where

$$\theta(y)=\begin{cases}1 & y\geqslant 0\\ 0 & y<0\end{cases}, \qquad v_g=\frac{\partial\varepsilon_g^0}{\partial g}.$$

Integrating with respect to x, we obtain

$$\Gamma_1 \sim \frac{\lambda_0}{p_0 q}\left[\ln\left|\frac{v_g q+\omega}{v_g q-\omega}\right| - i\pi\theta(v_g q-|\omega|)\right] \times \int_{\varepsilon-\mu_0-\omega_0/2}^{\varepsilon+\omega_0/2-\mu_0}\left[\theta\left(t+\frac{\omega}{2}\right)-\theta\left(t-\frac{\omega}{2}\right)\right]dt. \qquad (9)$$

The last integral differs from zero in the region $|\epsilon-\mu_0| \lesssim \omega_0$ and is of the order of ω_1, where ω_1 is the smaller of the numbers ω_0 and ω. It follows from (9) that the largest value $\Gamma_1 \sim \lambda_0$ is reached for $\omega \sim \omega_0$ and $q \sim \omega_0/p_0$. These values of q and ω play no part in our subsequent calculations. Indeed, for the calculation of $\Sigma(p)$ according to (5) the essential values are $q \sim p_0$ and $\omega \sim \omega_0$, for which it follows from (8) that $\Gamma_1 \sim M^{-1/2}$. For obtaining $\Pi(q)$ the essential values are $\omega \sim \omega_q \sim qp_0/\sqrt{M} \ll p_0 q$ (ω_q is the frequency of a phonon of momentum q). From (9) we obtain

$$\Gamma_1 \sim \lambda_0 \frac{\omega}{p_0 q}\left[\frac{\omega}{2v_g q}-i\pi\right] \sim \frac{\lambda_0}{M}+i\frac{\lambda_0}{\sqrt{M}}.$$

We thus have $\Gamma = 1 + O(M^{-1/2})$. It can be shown that this estimate is not changed when diagrams of a higher order are taken into account.

4. THE PHONON GREEN'S FUNCTION

As will be seen from our subsequent calculations, $G(\mathbf{p}, \epsilon)$ differs essentially from $G_0(\mathbf{p}, \epsilon)$ only in a narrow range of values of p and ϵ: $|\mathbf{p}-\mathbf{p}_0| \sim \omega_0/p_0$; $\epsilon - \epsilon_0^0 \sim \omega_0$. In the calculation of $\Pi(q, \omega)$ according to (5) the integration is performed over a wide range of the variables, which permits us to replace G by G_0 accurately to terms $\sim M^{-1/2}$. For $\Pi(\mathbf{q}, \omega)$ we obtain from (5)

$$\Pi(\mathbf{q},\omega) \approx \frac{1}{i}\int G_0\left(p+\frac{q}{2}\right)G_0\left(p-\frac{q}{2}\right)d^4p. \qquad (10)$$

Integration of (10) with respect to ϵ gives

$$\Pi(\mathbf{q},\omega)=\frac{1}{(2\pi)^3}\int\frac{n(\mathbf{p}-\mathbf{q}/2)-n(\mathbf{p}+\mathbf{q}/2)}{\varepsilon^0_{\mathbf{p}+\mathbf{q}/2}-\varepsilon^0_{\mathbf{p}-\mathbf{q}/2}-\omega-i\delta\omega/|\omega|}d^3p, \qquad (11)$$

where

$$n(p)=\begin{cases}1 & p>p_0\\ 0 & p<p_0.\end{cases}$$

We see from (10) and (11) that $\Pi(\mathbf{q}, \omega)$ is an even function of ω. For subsequent calculations the important values of $\Pi(\mathbf{q}, \omega)$ are obtained for

$$\omega \sim \omega_q \sim p_0 q/\sqrt{M} \ll p_0 q.$$

With these values of ω we have from (11)

$$\Pi(\mathbf{q},\omega)=\frac{p_0}{(2\pi)^2}\left[g\left(\frac{q}{2p_0}\right)+\pi i\frac{|\omega|}{2p_0 q}\right], \qquad (12)$$

where

$$g(x)=\frac{1}{2}\left[1+\frac{1-x^2}{2x}\ln\left|\frac{1+x}{1-x}\right|\right]. \qquad (13)$$

g(x) can be represented in the interval $0 < x < 1$ with sufficient accuracy by

$$g(x) \approx 1 - x^2/2. \qquad (13')$$

From (2), (5) and (12) we obtain

$$D(\mathbf{q},\omega)=\frac{1}{D_0^{-1}(\mathbf{q},\omega)-\Pi(\mathbf{q},\omega)} = \frac{2\omega_q^0\alpha_q^2}{\left[(\omega_q^0)^2-\omega^2-(\omega_q^0)^2\lambda_0\left(g\left(\frac{q}{2p_0}\right)+i\pi\frac{|\omega|}{2p_0 q}\right)\right]}.$$

The real part of the pole of $D(\mathbf{q}, \omega)$ gives the renormalized phonon frequency

$$\omega_q^2=(\omega_q^0)^2\left[1-\lambda_0 g\left(\frac{q}{2p_0}\right)\right] \approx (\omega_q^0)^2\left(1-\lambda_0+\lambda_0\frac{q^2}{8p_0^2}\right). \qquad (14)$$

Eq. (13′) was used in the derivation of this last equation. The imaginary part of the pole gives the phonon attenuation

$$\delta_1(q) = {}^1/_4\pi\lambda_0(\omega_q^0)^2/p_0 q. \qquad (15)$$

The relative attenuation is given by

$$\delta_1(q)/\omega_q = \lambda_0(\omega_q^0)^2\pi/4p_0 q\omega_q \sim \lambda_0/\sqrt{M} \ll 1.$$

From (14) and (15)

$$D(\mathbf{q},\omega)=\alpha_q^2\frac{\omega_q^0}{\omega_q}\left(\frac{1}{\omega_q-\omega-i\delta_1(q)\omega/|\omega|} + \frac{1}{\omega_q+\omega-i\delta_1(q)\omega/|\omega|}\right). \qquad (16)$$

Thus the phonon Green's function D, which was obtained by taking the interaction with electrons into account, differs from D_0 through replacement of the frequencies ω_q^0 by $\omega_q - i\delta_1(q)$ and the occurrence of the renormalizing factor ω_q^0/ω_q.

5. THE ELECTRON GREEN'S FUNCTION

From (5) we have

$$G = 1/[\varepsilon_p^0 - \varepsilon - \Sigma(\mathbf{p},\varepsilon)], \tag{17}$$

$$\Sigma(\mathbf{p},\varepsilon) = \frac{1}{i(2\pi)^4} \int\limits_{|\mathbf{p}-\mathbf{p}_1|<q_m} D(\mathbf{p}-\mathbf{p}_1, \varepsilon-\varepsilon_1)\, d\mathbf{p}_1\, d\varepsilon_1/[\varepsilon_{p_1}^0 - \varepsilon_1 - \Sigma(\mathbf{p}_1, \varepsilon_1)]. \tag{18}$$

G will now be obtained by solving the integral equation (18).

It is easily seen that $\Sigma(\mathbf{p}, \epsilon) \sim \omega_0$, so that G differs essentially from G_0 only for $|\epsilon_p^0 - \epsilon| \sim \omega_0$. The relative change of the excitation energy is large only for $\epsilon_p^0 - \mu_0 \sim \omega_0$. Thus the electron excitation spectrum varies appreciably only close to the Fermi surface in the range $p - p_0 \sim \omega_0/p_0$.

We introduce the notation

$$\begin{aligned} &(\varepsilon_p^0 - \mu_0)/\omega_0 = \xi,\ (\varepsilon-\mu)/\omega_0 = \eta, \\ &\Sigma(\xi,\eta) = \Sigma(0,0) + \omega_0 f(\xi,\eta), \end{aligned} \tag{19}$$

where

$$\begin{aligned} &\mu = \mu_0 + \Sigma(0,0), \\ &\omega_0 = \omega_q|_{q=2p_0} = \omega_{2p_0}^0\left(1 - \frac{\lambda_0}{2}\right). \end{aligned} \tag{19'}$$

As was shown in reference 3, the imaginary part of $\Sigma(\xi, \eta)$ must vanish for any value of ξ when ϵ equals the chemical potential.

As will be shown below $\Sigma(0, 0)$ is real. Therefore

$$\operatorname{Im}\Sigma(0,\eta) = \omega_0 \operatorname{Im} f(0,\eta);$$

since according to (19)

$$f(0,\eta)|_{\eta=0} = 0$$

and thus Im Σ vanishes for $\epsilon = \mu$ (μ is the chemical potential).

In the notation of (19) and (19′) G becomes

$$G(\xi,\eta) = \frac{1}{\omega_0}\frac{1}{\xi - \eta + f(\xi,\eta)}. \tag{20}$$

From the foregoing discussion we are interested in ξ and $\eta \sim 1$.

In (19) we pass from integration over the angles of the vector $\mathbf{p}_1$ to integration over $q = |\mathbf{p}-\mathbf{p}_1|$: $q\,dq = pp_1\,dx$, where x is the cosine of the angle between $\mathbf{p}$ and $\mathbf{p}_1$. We have

$$\begin{aligned} \Sigma(\mathbf{p},\varepsilon) = \frac{1}{i(2\pi)^3 p}\int\limits_{-\infty}^{\infty} d\omega \int\limits_{|p-p_1|<q<p+p_1;\ q<q_m} q\,dq\,p_1\,dp_1 \\ \times D(q,\omega)/[\varepsilon_{p_1}^0 - \varepsilon - \omega - \Sigma(p_1, \varepsilon+\omega)], \end{aligned} \tag{21}$$

or

$$\begin{aligned} \Sigma(p,\varepsilon) = \frac{1}{(2\pi)^3 ip}\int\limits_{-\infty}^{\infty} d\omega \int\limits_0^{q_m} q\,dq D(q,\omega) \\ \times \int\limits_{|p-q|}^{p+q} p_1\,dp_1/[\varepsilon_{p_1}^0 - \varepsilon - \omega - \Sigma(p_1, \varepsilon+\omega)]. \end{aligned} \tag{21'}$$

As mentioned previously, values of p close to p_0 are of interest, so that $(p-p_0)/p_0 \sim 1/\sqrt{M}$. Therefore in the right-hand side of (21) p can be replaced by p_0 accurately to within $M^{-1/2}$. We divide the integration over p_1 into two regions defined by

$$1)\ |\xi_1| = |\varepsilon_{p_1}^0 - \mu_0|/\omega_0 \leqslant \gamma \text{ and } 2)\ |\xi_1| > \gamma,$$

where γ lies between the following limits:

$$1 \ll \gamma \ll 1/\nu,\quad \nu = \omega_0/p_0 \sim 1/\sqrt{M}. \tag{22}$$

In the integral over region 1 $\Sigma(p, \epsilon)$ in the integrand can be replaced by $\Sigma(p_0, \epsilon+\omega)$ accurately to within $\sim M^{-1/2}$.

We note that for $q_m > 2p_0$ region 1 exists only for $q < 2p_0$. Therefore for the integration over p_1 in (21), in the term corresponding to region 1 the integration over q is carried as far as the smaller of the quantities q_m and $2p_0$.

Since $D(q, \omega)$ for $\omega \gg \omega_0$ vanishes as ω^{-2}, in the integral of (21) the essential result is found for $\omega \sim \omega_0$, and for integration in region 2 we can neglect $\epsilon - \mu_0$ and $\Sigma(p_1, \epsilon+\omega)$ in the denominator of the integrand compared with $\epsilon_{p_1}^0 - \mu_0$, with accuracy $\sim 1/\gamma$. Therefore integration over the region $|\xi_1| > \gamma$ introduces into Σ a term which is independent of p and ϵ. This term is $\Sigma(0, 0) = \mu - \mu_0$, since integration over the region $|\xi_1| < \gamma$ yields an expression which vanishes for $\eta = 0$.

Therefore the change of the chemical potential is given by

$$\begin{aligned} \mu - \mu_0 = \Sigma(0,0) \\ = \frac{1}{ip(2\pi)^3}\int\limits_{-\infty}^{\infty} d\omega \int\limits_0^{q_m} q\,dqD(q,\omega) \int\limits_{|p_0-q|}^{p_0+q} \frac{p_1\,dp_1}{\varepsilon_{p_1}^0 - \mu_0}. \end{aligned} \tag{23}$$

The integral over p_1 in (23) is taken in the sense of the principal value, which corresponds to dropping of the region $|\xi_1| < \gamma$, which is small compared with the essential region of integration.

Subtracting $\Sigma(0, 0)$ from the left and right members of (21) and dividing by ω_0, we obtain an integral equation for $f(0, \eta) = f(\eta)$:

$$f(\eta) = \frac{1}{ip(2\pi)^3}\int\limits_{-\infty}^{\infty} d\eta' \int\limits_{-\gamma}^{\gamma} d\xi_1 \int\limits_0^{q_1} \frac{D(q, \omega_0\eta')\,q\,dq}{\xi_1 - \eta - \eta' - f(\eta+\eta')}, \tag{24}$$

where q_1 is the smaller of the numbers q_m and $2p_0$. Here terms $\nu\xi$, $\nu\xi_1 \ll 1$ have been dropped.

Since the essential values in (24) are $\eta' \ll \gamma$ the limits with respect to ξ_1 can be replaced by $\pm\infty$.

Integration over ξ_1 gives

$$\lim_{\gamma\to\infty}\int_{-\gamma}^{\gamma}\frac{d\xi_1}{\xi_1-\varphi(\eta,\eta')}=\pi i\,\mathrm{Sgn}\,\mathrm{Im}\,\varphi(\eta,\eta')=\pi i\,\mathrm{Sgn}\,f_1(\eta+\eta'),$$

where f_1 denotes the imaginary part of f. As was shown in reference 2, the imaginary part of the Green's function, and thus $f_1(\eta)$, reverses its sign for $\eta = 0$; moreover, $f_1(\eta) > 0$ for $\eta > 0$. Therefore

$$\mathrm{Sgn}\,f_1(\eta)=\mathrm{Sgn}\,\eta.$$

Inserting these results in (24), we obtain

$$f(\eta)=\frac{1}{8\pi^2p_0}\int_0^{q_1}q\,dq\int_0^{\infty}\mathrm{Sgn}\,(\eta+\eta')\,D(q,\omega_0\eta')\,d\eta'$$
$$=\frac{1}{8\pi^2p_0}\int_0^{q_1}q\,dq\int_{-\eta}^{\eta}D(q,\omega_0\eta')\,d\eta'.$$

We use the notation

$$f(\eta)=f_0(\eta)+if_1(\eta).$$

For f_0 and f_1 we obtain

$$f_0=\frac{\alpha_q^2}{8\pi^2p_0\omega_q^0}\int_0^{q_1}q\,dq\int_{-\eta}^{\eta}(\omega_q^0)^2\,\frac{2}{\omega_q^2-\omega_0^2\eta'^2}\,d\eta', \quad (25a)$$

$$f_1=\frac{\alpha_q^2}{8\pi^2p_0\omega_q^0}\,2\pi\int_0^{q_1}(\omega_q^0)^2\,q\,dq\int_{-\eta}^{\eta}\frac{2\delta_1(q)\,|\eta'|\,\omega_0}{(\omega_q^2-\omega_0^2\eta'^2)^2+4\delta_1^2\omega_0^2\eta'^2}\,\frac{d\eta'}{\pi}. \quad (25b)$$

It is easily seen that for $\eta \gg 1/\sqrt{M}$ the integrand with respect to η' in (25b) can be replaced by $\delta(\omega_q - \omega_0\eta')$. Using the notation $x = q/2p_0$ and integrating over η', we obtain

$$f_0=\lambda_0\int_0^{x_1}x\,dx\,\frac{(\omega_q^0)^2}{\omega_0\omega_q}\ln\left|\frac{\eta+\omega_q/\omega_0}{\eta-\omega_q/\omega_0}\right|,$$
$$f_1=\pi\lambda_0\int_0^{y}x\,dx\,\frac{(\omega_q^0)^2}{\omega_0\omega_q}. \quad (26)$$

Here $x_1 = q_1/2p_0$, $y = g/2p_0$, where g is given by the condition $\omega_g = \omega_0\eta$ for $|\eta| < 1$ and $y = x_1$ for $|\eta| > 1$.

We introduce the variable $t = \omega_q/\omega_0$. According to (14) and (19'), t is related to x by

$$t^2=\frac{1}{1-\lambda_0/2}x^2\left(1-\lambda_0+\frac{\lambda_0}{2}x^2\right).$$

For f_0 we obtain

$$f_0=\int_0^1 dt\,\ln\left|\frac{t+\eta}{t-\eta}\right|\left(1-\frac{a}{\sqrt{a^2+t^2}}\right),\quad a^2=\frac{(1-\lambda_0)^2}{\lambda_0(2-\lambda_0)}, \quad (27)$$

For $\eta \ll 1$ we have

$$f_0(\eta)=2\eta\ln\frac{1-\lambda_0/2}{1-\lambda_0}=\lambda\eta. \quad (28)$$

The imaginary part, $f_1(\eta)$, is

$$f_1(\eta)=\pi\int_0^{t_1}\left[1-\frac{a}{\sqrt{a^2+t^2}}\right]dt=\pi\left\{t_1-a\ln\frac{t_1+\sqrt{a^2+t_1^2}}{a}\right\}, \quad (29)$$

where $t_1 = \eta$ for $|\eta| < 1$ and $t_1 = 1$ for $|\eta| > 1$; for $M^{-1/2} \ll \eta \ll 1$ we obtain

$$f_1(\eta)=\frac{\pi\eta^3}{6a^2}=\frac{\pi\lambda_0(2-\lambda_0)}{6(1-\lambda_0)^2}\eta^3. \quad (30)$$

For $\eta \ll 1/\sqrt{M}$ in the denominator of the integrand of (25b) η' can be neglected. This gives

$$f_1=\lambda_0^2\frac{\omega_0}{4p_0^2}\int_0^{x_1}\frac{dx}{[1-\lambda_0 g(x)]^2}\,\eta\,|\eta|. \quad (31)$$

The attenuation of the electron excitations given by (30) results from the emission of phonons. When the energy of a quasi-particle is very close to the Fermi surface ($\eta \ll M^{-1/2}$), a different attenuation mechanism is more important; this is attenuation due to the interaction between electrons, which results from phonon exchange. As mentioned above, interelectronic interaction leads to attenuation which is proportional to the square of the short distance from the Fermi surface, as follows from (31).

The electron energy spectrum is determined by the poles of G, that is, by the condition

$$\eta+f(\eta)=\xi. \quad (32)$$

For small η we have

$$\eta(\xi)\approx\xi/(1+\lambda).$$

Returning to the usual notation and subtracting the energy of a hole, we obtain for the excitation energy

$$E_{p_1p_2}=\varepsilon_{p_2}-\varepsilon_{p_1}=(\varepsilon_{p_2}^0-\varepsilon_{p_1}^0)/(1+\lambda)$$
$$=v_0^0(p_2-p_1)/(1+\lambda)=v_0(p_2-p_1), \quad (33)$$

where $p_2 > p_0$, $p_1 < p_0$ and v_0^0 is the unrenormalized velocity on the Fermi surface. Renormalization of the velocity on the Fermi surface is given by

$$v_0=\frac{v_0^0}{1+\lambda};\quad \lambda=2\ln\frac{1-\lambda_0/2}{1-\lambda_0}. \quad (34)$$

For $\lambda_0 \ll 1$ we obtain from (34)

$$v_0=v_0^0(1-\lambda_0).$$

This equation agrees with the result that Fröhlich obtained by using perturbation theory.

Equation (34) shows that $v_0 > 0$ for all values of λ, and the rearrangement of the Fermi distribution which Fröhlich predicted does not occur.

It follows from (30) that for $\lambda_0 \sim 1$ the excitation attenuation equals the excitation energy in order of magnitude for $\eta \sim 1$, i.e., for the excitation energy

$$E_{p_1 p_2} \sim \omega_0.$$

With further increase of the excitation energy, the attenuation ceases to increase and becomes smaller than the excitation energy. Thus for $\lambda_0 \sim 1$ electron excitations in the region $E_{p_1 p_2} \sim \omega_0$ cannot be described by means of quasi-particles.

[1] L. N. Cooper, Phys. Rev. **104**, 1189 (1956); Bardeen, Cooper and Schrieffer, Phys. Rev. **106**, 162 (1957); N. N. Bogoliubov, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 58 (1958), Soviet Phys. JETP **7**, 41 (1958); L. P. Gor'kov, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 735 (1958), Soviet Phys. JETP **7**, 505 (1958).

[2] H. Fröhlich, Phys. Rev. **79**, 845 (1950).

[3] V. M. Galitskii and A. B. Migdal, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 139 (1958), Soviet Phys. JETP **7**, 96 (1958).

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COOLING OF AIR BY RADIATION

II. STRONG COOLING WAVE

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Radiation-cooling wave of air accompanied by a large temperature drop, is considered. It is shown that, the radiation is always from the lower edge of the wave, regardless of the value of the upper temperature, and that the radiation transfer inside a strong wave has the character of radiant heat conduction. The strong-wave mode with adiabatic cooling is considered.

IN the first part of this work (reference 1)* we have described qualitatively the cooling of a large volume of hot air by radiation. We have found in this case that a unique temperature profile is developed in the air in the form of a step or a cooling wave (CW) propagating towards the hotter air. The air in the wave cools down from a high temperature T_1 to a lower temperature T_2. At the lower temperature T_2 the air becomes transparent, i.e., stops absorbing and emitting radiation.

In reference 1 we have considered the limiting case of a weak CW, in which the upper and lower temperatures T_1 and T_2 are not greatly different, and consequently the flux from the CW is close enough to either σT_1^4 or to σT_2^4. In this article we present the theory of a strong CW, in which the upper temperature can be unlimited. The fundamental problem consists obviously of determining the radiation flux from the front of the CW to infinity. Another problem is to find the temperature distribution in the front of the CW.

1. DETERMINATION OF THE RADIATION FLUX FROM THE FRONT OF THE COOLING WAVE

It was indicated in reference 1 that to find the stationary mode of the CW it is necessary to employ one of two procedures. In the first we introduce a constant adiabatic-cooling term into the energy equation. In the second we determine at the very outset the transparency temperature T_2, using formula (I.4). We then assume that when

*Henceforth, when referring to the formulas of the first part of this article, we shall precede the number of the formula by I [e.g. (I.4), (I.10)].