THE RELATIVISTIC THEORY OF REACTIONS INVOLVING POLARIZED PARTICLES

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It is shown that, in the rest system, the relativistic formulas for the angular distribution and the polarization vectors and tensors for a reaction of the type $a + b \rightarrow c + d$ are essentially the same as the nonrelativistic formulas, if the spin of a particle is defined as its internal angular momentum around its center of mass. The square of this internal angular momentum is Lorentz invariant. The spins of the particles are arbitrary, and their rest masses are nonvanishing.

The main difference from the nonrelativistic case is that the description of the spin state is not the same in different Lorentz reference systems. Therefore for cascades of reactions (for example, for experiments on double scattering) corrections must be applied to the nonrelativistic formal theory. The relativistic changes in the angular correlations are indicated for successive reactions of the type $\pi + p \rightarrow Y + K$, $Y \rightarrow N + \pi$.

INTRODUCTION

FOR reactions of the type $a + b \rightarrow c + d$, formal theories are known which express the angular distribution and the state of polarization of the products of a reaction in terms of the states of polarization of the incident beam and the target and unknown parameters which are the elements of the S matrix for the process $a + b \rightarrow c + d$. The simplest example is the well-known formula for the function $f(\theta)$ which appears in the expression

$$\psi(r) = e^{ikr} + f(\theta)\, e^{ikr}/r$$

for the wave function of a stationary scattering process of particles of spin zero. In this case the unknown parameters are called the scattering phase shifts.

These theories are based on the use of the conservation laws (in particular the law of the conservation of the total angular momentum). Coester and Jauch[1] were the first to obtain formulas for the angular distribution and polarization in the case of particles a, b, c, d of arbitrary spins; their starting point was the explicit formulation of the conservation laws in terms of the diagonal property of the S matrix with respect to the conserved quantities. These same formulas have been obtained by Simon and Welton, but by a different method (cf., e.g., Ref. 2).

These formulas are nonrelativistic, but only because the spin state of the particles is described in the Pauli approximation (so that it has the same appearance in all Lorentz reference systems). The theory of the scattering of spinless particles is essentially relativistic. In order to obtain the angular distribution in any Lorentz system, one has only to transform $\sigma(\theta) = |f(\theta)|^2$ from the center-of-mass system into the desired system by using known formulas. What is therefore required for the relativistic generalization is the definition of the relativistic spin operator. The spin operator which we introduce satisfies all the requirements which can be demanded in terms of the concept of the spin as the intrinsic angular momentum of a particle.*

To obtain the relativistic formulas we use the method of Coester and Jauch in a form presented in a paper by one of us.[3] We emphasize that in this method one needs only the ability to describe the state of a free particle possessing spin; we do not need relativistic equations (for free particles) like the Dirac equation (which plays an essential part in Stapp's relativistic theory[4] of the scattering of particles with spin $\frac{1}{2}$).

1. THE CONSERVED PHYSICAL QUANTITIES IN A RELATIVISTIC THEORY

The conservation laws are simply an expression of the fact that the physical processes in an isolated system must not depend on the means used to de-

*Iu. M. Shirokov has informed us that he has employed this same description of the spin state (which was obtained by him from the theory of the irreducible representations of the inhomogeneous Lorentz group) to the formulation of a similar relativistic theory of polarization and correlation effects.

scribe it, in particular on the choice of the reference system. Here it is of course assumed that space-time is homogeneous and isotropic (we can suppose, however, that this assumption is contained in the concept of an isolated system). In quantum mechanics this fact is expressed by the requirement that the S matrix of a physical process must commute with ten operators, the infinitesimal displacements of the origin of the space and time coordinates providing operators P_μ, and infinitesimal space-time rotations giving operators $M_{\mu\nu}$. The fact that an operator commutes with the S matrix means that the S matrix is diagonal with respect to the eigenvalues of this operator,* and consequently that the corresponding physical quantity is conserved, i.e., remains unchanged in all internal processes.

Four conservation laws have the clear physical meaning of the conservation of the total momentum and energy $P_\mu \{P_x, P_y, P_z, iP_0\}$. Of the six other operators $M_{\mu\nu}$, three operators M_{k4} ($k = 1, 2, 3$) do not have an immediate physical meaning, and instead of $M_{\mu\nu}$ we shall introduce six other operators, which have the physical meaning of the coordinates of the center of mass of the physical system and its total angular momentum around its center of mass.

The properties of the center of mass follow directly from its conceptual meaning: the motion of the system as a whole can be characterized primarily (in the very first approximation) as the motion of a material point with mass equal to the rest mass (or energy) of the system and with momentum equal to the total momentum **P** of the system. The center of mass of an isolated system must therefore move uniformly in a straight line. Furthermore, in quantum mechanics we must require that the center of mass **R** actually be the coordinate operator of a certain particle, i.e., in particular, that the well known commutation relations hold between R_x, R_y, R_z, P_x, P_y, P_z.

We can obtain such an operator **R** in the following way. It is well known that the following are the commutation relations which must be satisfied by the operators P_i and $M_{\mu\nu}$ (cf. Ref. 6 and Sec. 3 of Ref. 5):

$$[P_i, P_j] = 0;\ [P_i, E] = 0;\ [M_i, P_j] = i\varepsilon_{ijk}P_k; \quad (1.1)$$

$$[M_i, E] = 0;\ [M_i, M_j] = i\varepsilon_{ijk}M_k; \quad (1.2)$$

$$[N_i, P_j] = i\delta_{ij}E;\ [N_i, E] = iP_i;\ [M_i, N_j] = i\varepsilon_{ijk}N_k; \quad (1.3)$$

$$[N_i, N_j] = -i\varepsilon_{ijk}M_k. \quad (1.4)$$

Notations: [A, B] = AB − BA; i, j, k take the values 1, 2, 3; and

$$\{M_1, M_2, M_3\} = \{M_{23}, M_{31}, M_{12}\};\ iN_j = M_{j4};$$

$$E = (\mathbf{P}^2 + m_0^2)^{1/2};\ \hbar = c = 1;$$

ϵ_{ijk} is a tensor antisymmetric in all its indices, with $\epsilon_{123} = 1$. It is understood that we confine ourselves to those state vectors ψ_0 which describe states with a definite rest mass m_0, i.e., for which $P_\mu P_\mu \psi_0 = -m_0^2\psi_0$, or $(P_0 - E)\psi_0 = 0$ (the time displacement operator P_0 is equivalent to the factor E). We note that since $[N_i, E] = iP_i$ the average value of N_i is a linear function of the time. Therefore N_i is "conserved" in the sense that internal processes have no effect on this time dependence.

We introduce three new operators R_x, R_y, R_z, for which

$$[R_i, R_j] = 0, \quad [R_i, P_j] = i\delta_{ij}$$

(from which if follows that $[R_i, E] = iP_i/E$). We represent **M** in the form

$$M_k = \sum_{i,j} \varepsilon_{kij} R_i P_j + J_k,$$

and find from Eq. (1.1) that $[J_i, P_j] = 0$. If we require that R_x, R_y, R_z be the components of a spatial vector (as must indeed be the case), i.e., that $[M_i, R_j] = i\epsilon_{ijk}R_k$, then $[J_i, R_j] = 0$, and J_x, J_y, J_z also form a three-dimensional vector. It then follows from Eq. (1.2) that $[J_i, J_j] = i\epsilon_{ijk}J_k$. In like fashion, representing **N** in the form

$$N_i = {}^1/_2 (R_iE + ER_i) + K_i \equiv R_iE - iP_i/2E + K_i,$$

we find that $[K_i, P_j] = 0$. Therefore also $[K_i, E] = 0$, and the average value of K_i is constant in time.

We now pose the problem of expressing **M** and **N** in terms of the operators **R** and **J** which we have introduced. For **M** this has already been done. It remains only to express the spatial polar vector **K** in terms of **P** and **J**. It can be shown that if R_x, R_y, R_z are the first three components of any four-vector (i.e., for example, $[N, R] = 0$ if $i \neq j$), then **K** cannot be expressed in terms of **J** and **P** only so as to satisfy all the commutation relations for **N**. This means that if **K** is constructed from **P** and **J** alone, then $[\mathbf{R}, \mathbf{P}]_k$ and J_k are not the spatial components of a four-dimensional tensor of the second rank.

*Let us write out AS − SA = 0 as a matrix product. In doing so, we choose a representation in which the operator A is diagonal (i.e., we label matrix elements with its eigenvalues). Then

$$A_{ik}S_{kl} - S_{im}A_{ml} = (A_i - A_l)S_{il} = 0,$$

i.e., S_{il} must be equal to zero if $i \neq l$.

The simplest $\mathbf{K}$ (namely one linear in J_x, J_y, J_z) satisfying Eqs. (1.3) and (1.4) has the form

$$\mathbf{K} = [\mathbf{PJ}](E \pm m)^{-1}$$

(cf. Refs. 7 and 6).*

Since the problem has a solution, it follows that: (1) $\mathbf{R}$ is "conserved" (in the same sence as $\mathbf{N}$), since it can be expressed in terms of the conserved operators $M_{\mu\nu}$ (see Appendix). $\mathbf{R}$ can be called the center-of-mass operator. It is the same as definition (e) of the center-of-mass in the papers of Pryce.[8] (2) $\mathbf{J}$ is also conserved and, what is particularly important for our purpose, $J^2 = J_x^2 + J_y^2 + J_z^2$ is Lorentz invariant, since $[\mathbf{N}, J^2] = 0$. We emphasize that this is true for arbitrary $\mathbf{K} = \mathbf{K}(\mathbf{P}, \mathbf{J})$.

2. USE OF THE CONSERVATION LAWS. RELATIVISTIC DEFINITION OF THE SPIN OF A PARTICLE

The four conservation laws for the total momentum and energy and the three conservation laws for the center of mass can be expressed very simply. The argument is usually carried through in the Lorentz system of reference K_S in which the (conserved) total momentum is zero (the so-called center-of-mass system). The origin of the coordinate system can be taken at the center of mass (more precisely, at the point given by the average value of the operator $\mathbf{R}$† for the particles a and b (or c and d). Then $\mathbf{J}$ is the total angular momentum. Since the commutation relations between J_x, J_y, J_z are the same as for the total angular momentum or for the Pauli spin matrices, the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$ are respectively equal to $h^2J(J+1)$ and $M = J, J-1, \ldots, -J$.

The conservation law for $\mathbf{J}$ is expressed by the fact that the S matrix is diagonal with respect to the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$:

$$(\ldots J'M' \mid S \mid \ldots JM) = (\ldots \mid S^{JM} \mid \ldots)\,\delta_{J'J}\delta_{M'M} \qquad (2)$$

and also the fact that $(\ldots \mid S^{JM} \mid \ldots)$ is independent of the value of M, which follows from $[J_x, S] = 0$.

To find, for example, the angular distribution of particles c and d, we need to know the explicit expression of the elements of the S matrix in the representation of the particle momenta. In order to express these elements in terms of the elements (2), we must first of all enumerate the remaining variables of a complete set (denoted in Eq. (2) by dots), which commute with each other and with J^2 and J_z.

The initial and final states of the process a + b → c + d are states of systems consisting of two free noninteracting particles. From the meaning of the S matrix, its elements are the transition amplitudes between such states. Therefore to label the elements of the S matrix we must take as our variables a complete set of quantum mechanical quantities describing the free particles a and b or c and d. The total angular momentum $\mathbf{J}$ (in the system K_S) is represented in the form $\mathbf{J} = \mathbf{j}_1 + \mathbf{j}_2$, where $\mathbf{j}$ is the total angular momentum of a single particle in K_S.

The procedure stated in Sec. 1 for obtaining the conserved angular momentum relative to the center of mass can be applied to a system of arbitrary physical nature (for example, to a system of fields). One needs only to know a concrete representation of the operators P and M. Therefore it is natural to apply this procedure to an "elementary" particle, whose physical nature is in general unknown (by the definition of "elementary"). Besides the coordinates $\mathbf{r}$ of the center of mass and the momentum $\mathbf{p}$ we then get just one conserved external characteristic of the particle, its angular momentum $\mathbf{s}$ relative to its center of mass $\mathbf{r}$. In defining the spin s of a particle, we are only fixing precisely the concept of the spin as the intrinsic angular momentum of the particle.

Accordingly, $\mathbf{j} = [\mathbf{r} \times \mathbf{p}] + \mathbf{s}$, and in the system K_S, in which $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}$, we get

$$\begin{aligned}\mathbf{J} &= [\mathbf{r}_1 \times \mathbf{p}_1] + \mathbf{s}_1 + [\mathbf{r}_2 \times \mathbf{p}_2] + \mathbf{s}_2 \\ &= [(\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{p}] + \mathbf{s}_1 + \mathbf{s}_2 \equiv \mathbf{l} + \mathbf{s}_1 + \mathbf{s}_2.\end{aligned} \qquad (3)$$

In the system K_S we can now proceed in complete formal analogy with the nonrelativistic treatment to introduce the total spin operator $\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2$ (the square of which, however, is not a Lorentz-invariant quantity). The eigenfunctions of the square of this quantity, s^2, and its component s_z can be expressed in terms of the products $\psi_{i_1 m_1} \times \psi_{i_2 m_2}$ of the eigenfunctions of the squares and components of the operators $\mathbf{s}_1$ and $\mathbf{s}_2$ (the eigenvalues of s_1^2 are denoted by $h^2 i_1(i_1+1)$). Since

*We have not been able to show that no other $\mathbf{K}$'s exist. Beginning with different considerations, L. G. Zastavenko has evidently proved the uniqueness of $\mathbf{K}$. We are grateful to him for a discussion of this question.

†The conservation law for $\mathbf{R}$ means something more than the conservation of the average value. The requirement $[\mathbf{R}, S] = 0$ means that if the system is in a state with a definite value of $\mathbf{R}$ (we note that in the interaction representation the wave function for the external behavior of the system does not change with the time), internal processes in the system will not take it out of this state. This property does not get used explicitly, but an operator $\mathbf{R}$ of this kind is required for the definition of the conserved angular momentum $\mathbf{J}$ of the system (and of the spin of a particle, see below).

the commutation relations for $\mathbf{s}$, $\mathbf{s}_1$, $\mathbf{s}_2$ have the usual form, $[s_x, s_y] = is_z$ and so on, the coefficients in these expressions will be the well known Clebsch-Gordan coefficients $(i_1 i_2 m_1 m_2 \mid i_1 i_2 s m)$, which are also the transformation functions for the transformation from the representation in the variables i_1, i_2, m_1, m_2 into the i_1, i_2, s, m-representation (and inversely). A similar meaning attaches to the coefficient $(l s \mu m \mid l s J M)$.

We can now take as the variables denoted by dots in Eq. (2) s, l, and the absolute values of the momenta in K_S (or the total energy of the system, which in K_S is equal to an invariant, the rest mass of the system).

3. FORMULAS FOR THE CROSS-SECTION AND THE POLARIZATION VECTOR AND TENSORS. RELATIVISTIC ROTATION OF THE SPIN

We can now express the elements of the S matrix in the representation of the momenta of the particles and their spin components in terms of the elements $(p_c, s', l', J', M' \mid S \mid p_a, s, l, J, M)$. The transformation function from the representation in the variables p, s, l, J, M into the representation of the momenta and the spin components is the product of three transformation functions. We can write out the transformation as follows (cf. Ref. 9):

$$(p_c;\ m_c,\ m_d \mid S \mid p_a,\ m_a,\ m_b) = (\vartheta_c \varphi_c p_c \mid l' \mu' p_c) \times (i_c i_d m_c m_d \mid i_c i_d s' m')\,(l' s' \mu' m' \mid l' s' J M)\,(s' l' \mid S^{J, E(p_a)} \mid s l) \times (l s J M \mid l s \mu m)\,(i_a i_b s m \mid i_a i_b m_a m_b)\,(l \mu p_a \mid \vartheta_a \varphi_a p_a). \tag{4}$$

We have used Eq. (2) and the law of the conservation of the total energy. $\mathbf{p}_c$ and $\mathbf{p}_a$ are the momenta of particles c and d and a and b respectively in K; ϑ_c, φ_c, p_c, ϑ_a, φ_a, p_a are their spherical angles and absolute values. It is recalled that p_c is a function of p_a:

$$\sqrt{p_a^2 + \varkappa_a^2} + \sqrt{p_a^2 + \varkappa_b^2} = \sqrt{p_c^2 + \varkappa_c^2} + \sqrt{p_c^2 + \varkappa_d^2}.$$

Summation over the labels l', μ', s', m', J, M, l, μ, s, m is understood.

$$(\vartheta \varphi p \mid l \mu p_0) = 2\pi\hbar \sqrt{\frac{2R}{V}} \frac{i^{-l}}{p} Y_{l\mu}(\vartheta, \varphi)\,(p \mid p_0),$$

where $Y_{l\mu}(\vartheta, \varphi)$ is a spherical harmonic. For the other notations see Ref. 3 (in particular, Appendix II).

From the formula $\rho' = S \rho S^+$ we can now get the density matrix ρ' of the products of the reaction in the representation of their momenta and spin components (ρ is the density matrix for beam and target in the same representation). The problems of normalization and of deducing the cross-section in the system K_S are solved in just the same way as in the nonrelativistic case (cf. Ref. 3). One can also introduce in just the same way the statistical polarization tensors instead of the density matrices. All the formulas will have the same form as those of the nonrelativistic case.[3] The difference lies in the fact that the quantities m_a, m_b, m_c, m_d (or τ, ν), and also the total spin and other variables are relative to the system K_S. The same spin state has a different form in a different Lorentz system K (for example, in the laboratory system).

In order to learn how a spin state specified in K_S is described in K, we must find the transformation function from the representation in the eigenvalues of s^2 and s_z in the system K_S to the representation in the eigenvalues of $\tilde{s}^2$ and $\tilde{s}_z$, which are the square and component of the same operator, but in the system K. In view of the fact that s^2 is Lorentz invariant, the spin operator $\tilde{\mathbf{s}}$ is a vector rotated as compared with $\mathbf{s}$. Consequently, the transformation function is the same as is obtained in the solution of the problem of describing a given spin state in a rotated system of spatial axes:

$$(\tilde{m} \mid m) = D^{i}_{\tilde{m}m}(\Phi_2, \theta, \Phi_1) = e^{-i\tilde{m}\Phi_1} i^{m-\tilde{m}} P^{i}_{\tilde{m}m}(\cos\theta)\, e^{-im\Phi_1}; \tag{5}$$

The matrices $P^{i}_{mm}(\cos\theta)$ are defined in Ref. 10 [Eq. (22) on page 77; we note that the matrix P^{1}_{mn} written out explicitly on page 78 does not agree with Eq. (22) and is incorrect]. If the rotation is interpreted as a turning of a vector in a stationary coordinate system, then it consists of (a) a rotation of the vector around the z axis by the angle Φ_1, (b) a rotation around the y axis by the angle θ, and (c) a rotation around the z axis by the angle Φ_2. All these rotations are counterclockwise. In the Appendix we show how to find the axis of rotation and the rotation angle Ω of the spin vector for the transformation from K_S to K. For the transformation of the spin state of the reaction products from K_S to the laboratory system we get as the Eulerian angles of the rotation

$$\{\Phi_1,\ \theta,\ \Phi_2\} = \{+\varphi,\ \Omega,\ -\varphi\},$$
$$\sin\Omega = \frac{\beta v \sin\vartheta\,(1 + \gamma + \gamma_\beta + \gamma')}{(1+\gamma)(1+\gamma_\beta)(1+\gamma')}\gamma\gamma_\beta, \tag{6}$$

where

$$v = |\mathbf{p}|/\omega = \sqrt{\omega^2 - \varkappa^2}/\omega;\ \gamma = \omega/\varkappa;$$
$$\gamma_\beta = (1 - \beta^2)^{-1/2};\ \gamma' = \omega'/\varkappa;$$

ω' is the energy of a reaction product in the laboratory system, and ϑ and φ are the spherical

angles of its momentum $\mathbf{p}$ in K_S, measured in an axis system with the z axis parallel to $\boldsymbol{\beta}$ (the x and y axes are chosen arbitrarily).

In Stapp's[4] Eq. (48) for $\sin\Omega$ there is a mistake (or a missprint): it does not have the factors $\gamma\gamma_\beta$ [$\gamma^{(a)}\gamma^{(b)}$ in his notation]. If one repeats Stapp's calculations (in accordance with his arguments), the result is just the present Eq. (6). The rotation by the angle Ω must be applied counterclockwise around the vector $\boldsymbol{\beta}[\mathbf{p}]$, $\boldsymbol{\beta}$ being the velocity of the laboratory system relative to the center-of-mass system of the reaction (see Appendix).

This relativistic effect of rotation of the spin state of course does not show up at all in the transformation of the angular distribution into the laboratory system (since the angular distribution is a polarization tensor of rank zero). One needs only to carry out the ordinary (kinematic) relativistic transformation of the angles from K_S to the laboratory system. The nonrelativistic theory of the angular distribution in reactions with unpolarized beams and targets remains valid also in the relativistic domain (except for changes or increased precision in the meanings of the quantities involved in the formulas).

As for the polarization vector and the polarization tensors, they are not directly measured in the experiments. In order to measure the polarization vector for the product c of the reaction $a + b \to c + d$, we must scatter c from a target e and measure the asymmetry in the angular distribution of the scattered particles c. Then we obtain information about the polarization vector $\mathbf{P}'$ in the center-of-mass system K'_S of the reaction $c + e \to c + e$. The desired polarization vector is obtained from $\mathbf{P}'$ by a rotation. The angle of this rotation is found from Eq. (6). In fact, in the successive Lorentz transformations from K_S to the laboratory system (by means of the known velocity $\boldsymbol{\beta}$) and then to the system K′ (velocity $\boldsymbol{\beta}'$) a rotation occurs only in the first transformation, since the momentum $\mathbf{p}'_c$ of particle c in the laboratory system is parallel to $\boldsymbol{\beta}'$, so that $\sin\Omega_2 \sim \sin(\hat{\boldsymbol{\beta}}'\mathbf{p}'_c) = 0$. This question is analyzed in more detail in Ref. 4, and the treatment carried through there is valid for arbitrary spin.

Quite generally, the relativistic rotation of the spin is seen to be of importance only for the treatment of cascades of reactions. In the following section we shall deal with the problem of the relativistic changes of the angular correlations in successions of reactions of type $a + b \to c + d$, $c \to e + f$.

In conclusion we point out that in the change from K_S to the system K_0 in which a particle is at rest the description of the spin state does not undergo any changes, since here $\boldsymbol{\beta} \parallel \mathbf{p}$ and $\Omega = 0$. Therefore we can regard the quantities m_a, m_b, and so on as describing the spin states of the particles in their rest systems K_0. This interpretation is preferable to the preceding one: the spin states of particles are described by quantities whose definition does not depend on the system K_S, i.e., on what target the particle is interacting with, on what its energy is, or on the energy balance of a particular reaction.*

4. THE RELATIVISTIC ANGULAR CORRELATIONS IN CASCADES OF THE TYPE $a + b \to c + d$, $c \to e + f$

We shall consider first the cascade $\pi^- + p \to Y + K$, $Y \to N + \pi$, which is well known in the literature. If the first reaction occurs near threshold, the correlation in the angle γ between the directions of the incident π^- mesons and the decay nucleons provides a way of determining the spin j of the hyperon Y. This correlation can be found if one substitutes into the expression

$$F(\vartheta, \varphi) = \frac{w}{V 4\pi} \sum_{q=0,2\ldots}^{2j-1} (2q+1)^{-1/2} Q(j, q) \sum_{\nu=-q}^{q} Y_{q\nu}(\vartheta, \varphi)\, \rho(q, \nu) \tag{7}$$

for the angular distribution of the decay products of the hyperon in its rest system K_Y the concrete expression for the statistical polarization tensors $\rho(q, \nu)$ of the hyperon (cf. Refs. 12 and 13). In the center-of-mass system K_S of the reaction $\pi^- + p \to Y + K$ we have near threshold (the z axis is directed along the π^- meson beam)

$$\rho_s(q, \nu) \sim Q(j, q)\, \delta_{\nu,0}. \tag{8}$$

The nonrelativistic correlation in the angle γ is

*In connection with this interpretation, however, the following misunderstanding can arise. Since there is only one system K_0 in which a particle is at rest, in any reaction the m values mean always the same thing: the spin components in the rest systems. Consequently it might seem that no transformations of the spin state are actually necessary. The point is that if we are given the velocity $\mathbf{v}_{21}$ of a system K_2 relative to K_1 and the velocity $\mathbf{v}_{32}$ of K_3 relative to K_2, then the velocity $\mathbf{v}_{31}$ (which is of course a function of $\mathbf{v}_{21}$ and $\mathbf{v}_{32}$) turns out not to be parallel to $\mathbf{v}_{13}$ if $[\mathbf{v}_{21}\mathbf{v}_{32}] \neq 0$ (cf. Ref. 11, Sec. 22). The transformation from K_1 to K_3 must have the form of a Lorentz transformation with a spatial rotation [see Eq. (58) of Ref. 11]. If the particle was at rest in K_1, then in K_3 it has the velocity $\mathbf{v}_{13}$, and by using this velocity we can go over to a system K_4 in which the particle is again at rest. Calculations show that the product of the transformations from K_1 to K_3 and then on to K_4 has the form of a pure spatial rotation: $\mathbf{s}_{(4)} = D^{-1}\mathbf{s}_{(1)}$, if $D\mathbf{v}_{13} = -\mathbf{v}_{31}$ (the space axes of the Lorentz systems K_1, K_2, K_3, K_4 are of course assumed parallel).

obtained by simply substituting Eq. (8) into Eq. (7):

$$F_{nr}(\vartheta,\varphi)\sim\sum_{q=0}^{2j-1}(2q+1)^{-1/2}Q^2(j,q)Y_{q,0},(\vartheta,0) \quad (9)$$

$$\sim\sum_{q=0}^{2j-1}Q^2(j,q)P_q(\cos\gamma).$$

In actual fact one must substitute into Eq. (7) not the $\rho_S(q,\nu)$, but the statistical tensors of the hyperon referred to the system K_Y:

$$\rho(q,\nu)=\sum_{\nu'}D^q_{\nu,\nu'}(\varphi_c,\Omega(\vartheta_c),-\varphi_c)\rho_s(q,\nu')$$

$$=\sqrt{4\pi/(2q+1)}\,Y^*_{q,\nu}(\Omega,\varphi_c)\,Q(j,q). \quad (10)$$

Here φ_C and ϑ_C are the spherical angles of the emission of the hyperon in the system K_S. The angle Ω is determined by Eq. (6), since in the transformation from K_S to the laboratory system and then on into K_Y a rotation occurs only in the passage from K_S to the laboratory system. (We note that in the experiment $F(\vartheta,\varphi)$ is obtained by translating the measured distribution from the laboratory system into K_Y, and not by going from K_S to the rest system of the hyperon).

Substituting Eq. (10) into Eq. (7), we get

$$F_r(\vartheta,\varphi)\sim\sum_{q=0}^{2j-1}(2q+1)^{-1}Q^2(j,q)\sum_{\nu=-q}^{+q}Y_{q\nu}(\vartheta,\varphi)Y^*_{q\nu}(\Omega,\varphi_c)$$

$$=(1/4\pi)\sum_{q=0}^{2j-1}Q^2(j,q)P_q(\cos\gamma_r), \quad (11)$$

where γ_r is now the angle between the direction of emergence of the decay products and the direction $\{\Omega(\vartheta_C),\varphi_C\}$. Thus the expression for the correlation has its old form, if we change the definition of the angle γ.

In the experiment we are discussing Ω does not exceed 1.5°. If we construct the distribution in γ, choosing only cases with fixed $\vartheta_C\approx 90°$ and fixed φ_C, then the difference between the nonrelativistic and the relativistic correlations can amount to 3% for $j=3/2$ and 5% for $j=5/2$. In the actual experimental procedure, of course, all cases of the cascade are used in the construction of the correlation $F(\gamma)$. If Eq. (11) is just integrated over φ_C, the difference between the resulting correlation $F_r(\gamma,\Omega(\vartheta_C))$ and the nonrelativistic result (9) does not exceed 0.1% over the entire range of angles γ and ϑ_C (for $j\le 5/2$).

In the case of the cascade $K^- + p \to Y + \pi$, $Y \to N + \pi$, the angular correlation again does not contain any unknown parameters and depends only on the spin j of the hyperon, if the energies of the K^- particles do not exceed 20 or 30 Mev but are large enough (> 0.1 Mev) so that mesic atoms are not formed (for more details see Ref. 9). In the center-of-mass system of the reaction $K^- + p \to Y + \pi$, with the z axis parallel to the direction of $[\mathbf{n}_K\times\mathbf{n}_Y]$, where $\mathbf{n}_K$ is the direction of the incident beam of K particles,

$$\rho_s(q,\tau)\sim Q(j,q)\,\delta_{\tau,0}.$$

Relative to this same set of axes, but in the rest system K_Y of the hyperon

$$\rho(q,\tau)=\sum_{\tau'}D^q_{\tau\tau'}(0,\Omega,0)\rho_s(q,\tau')$$

$$=\sqrt{4\pi/(2q+1)}\,Y^*_{q,\tau}(\Omega,0)\,Q(j,q). \quad (12)$$

The difference between the relativistic correlation $F_r(\vartheta,\varphi)$ and the nonrelativistic function is basically the same: substituting Eq. (12) into Eq. (7), we get the correlation $F_r(\theta)$ of the angle θ between the direction $\mathbf{n}$ of the emission of the decay products and the vector obtained by rotating $\mathbf{n}_Y$ by the angle Ω around the vector $[\mathbf{n}_K\times\mathbf{n}_Y]$ (i.e., in the plane of the reaction). The nonrelativistic correlation had the same form, but θ was the angle between $\mathbf{n}$ and $\mathbf{n}_Y$.

The correlation proposed by Adair[13] (cf. also Ref. 9) admits of energies larger than those near threshold, but does not change when treated relativistically: for this correlation one uses cases in which the hyperons are emitted at small angles with the direction of the incident beam, and then $\Omega\approx 0$.

Since the most general case of the cascade $a + b \to c + d$, $c \to e + f$, in which all the spins are arbitrary and the correlation depends on unknown parameters, is of no practical interest, we simply note without proof that the nonrelativistic form of the angular correlation can be retained. To do this one finds for each case of the cascade, using the measured angles of emission of the particle c, a particular system of coordinates belonging to this case. The angles of emission of the products from the decay of c are calculated relative to this system. The distribution in these recalculated angles has the old, nonrelativistic, form (but one has, of course, changed the rule for constructing the angular correlation from the experimental data).

APPENDIX

1. Let us express $\mathbf{s}$ in terms of $M_{\mu\nu}$ and p_μ. Let κ be the rest mass of the particle and $\omega = (p^2+\kappa^2)^{1/2}$, and

$$\mathbf{M}=[\mathbf{r}\times\mathbf{p}]+\mathbf{s},$$
$$\mathbf{N}=\mathbf{r}\omega-i\mathbf{p}/2\omega+[\mathbf{p}\times\mathbf{s}]/(\omega+\varkappa). \quad (A.1)$$

The four-dimensional vector

$$\Gamma_\sigma = (1/2i)\,\varepsilon_{\mu\nu\sigma\lambda} M_{\mu\nu} p_\lambda$$

($\epsilon_{\mu\nu\sigma\lambda}$ is the completely antisymmetric tensor of the fourth rank with $\epsilon_{1234} = 1$) then has the form:

$$\Gamma = s\varkappa + (ps)\,p/(\omega + \varkappa), \quad \Gamma_4 = i\,(sp). \tag{A.2}$$

Noting that $(\Gamma p) = \omega\,(sp)$, we find from Eq. (A.2):

$$s = \Gamma/\varkappa - (\Gamma p)\,p/\varkappa\omega\,(\omega + \varkappa). \tag{A.3}$$

All these operator equations are to be understood as being in the momentum representation.

From the second of the relations (A.1) we now get

$$r\omega = N + ip/2\omega - [p \times \Gamma]/\varkappa\,(\omega + \varkappa). \tag{A.4}$$

2. The vector $\tilde{s}$ in the new system K, which moves relative to K_S with the velocity β (in units of the speed of light) can now be found in the following way. Substituting into the right and left members of the equations [cf. Ref. 11, Sec. 18, Eq. (25)]

$$\tilde{\Gamma} = \Gamma + \beta\,\{(\Gamma\beta)\,(\gamma_\beta - 1)\,\beta^{-2} - \gamma_\beta \Gamma_{4i}\},$$

$$\tilde{\Gamma}_{4i} = \gamma_\beta\,\{\Gamma_{4i} - (\beta\Gamma)\} \tag{A.5}$$

the expressions (A.2) for Γ and $\tilde{\Gamma}$ in terms of s, p_μ and $\tilde{s}$, $\tilde{p}_\mu$, respectively, and replacing the $\tilde{p}_\mu$ by their expressions in terms of p_μ (which have the same form (A.5)), we get the expressions for $\tilde{s}$ in terms of s. First of all we establish the fact that $\tilde{s}$ is a linear combination of the vectors s, β, and p. This means that the vector $\tilde{s}$ is obtained from s by a rotation around an axis perpendicular to β and p. There remains only the determination of the magnitude and sign of the angle of rotation around this axis. For this purpose we choose a convenient set of three space axes (it is obvious that the angle of rotation cannot depend on the choice of the axes): $z \parallel \beta$, $y \parallel [\beta \times p]$. A rotation of the vector around the y axis in the counterclockwise direction through the angle Ω must have the form

$$\tilde{s}_x = \cos\Omega s_x + \sin\Omega s_z \qquad \tilde{s}_z = -\sin\Omega s_x + \cos\Omega s_z. \tag{A.6}$$

Representing the expression for $\tilde{s}$ in terms of s (in the chosen set of axes) in the form (A.6), and finding the coefficient of s_Z in the expression for $\tilde{s}_X$ (as that having the simplest form), after cumbersome calculations we get the formula (6) of Sec. 3 for $\sin\Omega$.

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