STRONG GRAVITATIONAL WAVES IN FREE SPACE

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The results of Einstein and Rosen[1] are generalized to the case of two interacting gravitational waves.

STRONG gravitational waves were first considered by Einstein and Rosen,[1] who showed that the equations for the propagation of the strong waves, corresponding in the limit to weak transverse-transverse waves of the $g_{22} - g_{33}$ type, reduce to a linear equation of propagation in the case of a cylindrical wave. It can be shown that even in the more general case, strong gravitational waves corresponding in the limit to incident waves of the types $g_{22} - g_{33}$ and g_{23} show a very similar simplification.

We shall start with integrals of the form

$$-ds^2 = Adx_1^2 + Cdx_2^2 + 2Bdx_2dx_3 + Ddx_3^2 - Adx_4^2. \quad (1)$$

where A, B, C, and D depend only on x_1 and x_4; in the limit this corresponds to a plane wave.

Here the coordinates x_1 and x_4 are so chosen that $g_{11} = -g_{44}$, $g_{14} = 0$ (see Einstein and Rosen[1]). This leads to the following expressions for the Christoffel symbols of the second kind (writing $CD - B^2 \equiv \alpha$, and denoting by lower suffixes on the quantities A, B, C, and D their derivatives with respect to the corresponding coordinates)

$$\Gamma_{11}^1 = \Gamma_{44}^1 = \Gamma_{14}^4 = (2A)^{-1}A_1; \qquad \Gamma_{11}^4 = \Gamma_{14}^1 = \Gamma_{44}^4 = (2A)^{-1}A_4;$$
$$\Gamma_{22}^1 = -(2A)^{-1}C_1; \qquad \Gamma_{22}^4 = (2A)^{-1}C_4;$$
$$\Gamma_{33}^1 = -(2A)^{-1}D_1; \qquad \Gamma_{33}^4 = (2A)^{-1}D_4;$$
$$\Gamma_{23}^1 = -(2A)^{-1}B_1; \qquad \Gamma_{23}^4 = (2A)^{-1}B_4;$$
$$\Gamma_{12}^2 = (2\alpha)^{-1}(DC_1 - BB_1); \qquad \Gamma_{42}^2 = (2\alpha)^{-1}(DC_4 - BB_4);$$
$$\Gamma_{13}^3 = (2\alpha)^{-1}(CD_1 - BB_1); \qquad \Gamma_{43}^3 = (2\alpha)^{-1}(CD_4 - BB_4);$$
$$\Gamma_{12}^3 = (2\alpha)^{-1}(CB_1 - BC_1); \qquad \Gamma_{42}^3 = (2\alpha)^{-1}(CB_4 - BC_4);$$
$$\Gamma_{13}^2 = (2\alpha)^{-1}(DB_1 - BD_1); \qquad \Gamma_{43}^2 = (2\alpha)^{-1}(DB_4 - BD_4).$$

Except for the $\Gamma_{kl}^i = \Gamma_{lk}^i$ formed from the above symbols by interchanging their lower indices, the other Γ_{kl}^i are equal to zero.

The only components of the curvature tensor $R_{ik} = R_{ikl}^l$ which do not vanish identically are R_{11}, R_{14}, R_{44}, R_{22}, R_{33}, and R_{23}. This gives six equations for the four quantities A, B, C, and D. It must therefore be proved that there is no contradiction in the equations. This will be done for a particular choice of coordinate system. Here we shall only note that two conditions have already been imposed on the choice of coordinate system: $g_{11} = -g_{44}$ and $g_{14} = 0$. Two other conditions are contained in the two extra equations over and above the four which would be sufficient.

We shall first write down the equations $R_{ik} = 0$ for the components 2 and 3. After some simplification they take the following form:

$$C_{44} - C_{11} + (2\alpha)^{-1}[C_1\alpha_1 - C_4\alpha_4 + 2C(B_1^2 - C_1D_1 - B_4^2 + C_4D_4)] = 0; \quad (2)$$

$$D_{44} - D_{11} + (2\alpha)^{-1}[D_1\alpha_1 - D_4\alpha_4 + 2D(B_1^2 - C_1D_1 - B_4^2 + C_4D_4)] = 0; \quad (3)$$

$$B_{44} - B_{11} + (2\alpha)^{-1}[B_1\alpha_1 - B_4\alpha_4 + 2B(B_1^2 - C_1D_1 - B_4^2 + C_4D_4)] = 0. \quad (4)$$

Now multiply equation (2) by D/2, multiply (3) by C/2, and (4) by −B, and add. This gives a linear wave equation for $\sqrt{\alpha}$:

$$(\sqrt{\alpha})_{11} - (\sqrt{\alpha})_{44} = 0. \quad (5)$$

But, as Einstein and Rosen showed,[1] the form of the linear element of distance (1) remains invariant under all coordinate transformations $\bar{x}_1 = \bar{x}_1(x_1, x_4)$ and $\bar{x}_4 = \bar{x}_4(x_1, x_4)$ which satisfy the equation

$$(\bar{x}_1)_{11} - (\bar{x}_1)_{44} = 0. \quad (6)$$

Therefore it is possible, without any loss of generality, to assume

$$\sqrt{\alpha} = \bar{x}_1. \quad (7)$$

This can also be demonstrated in a different way, as follows. The general solution of (5) is $\sqrt{\alpha} = f(x_1 + x_4) + g(x_1 - x_4)$. Let us choose a coordinate system such that

$$f(x_1 + x_4) = (x_1 + x_4)/2, \quad g(x_1 - x_4) = (x_1 - x_4)/2.$$

This choice is compatible with the integral form (1) because of condition (6).

We shall now show that there is no contradiction in the entire system of gravitational equations for the special case $\sqrt{\alpha} = x_1$. (The bars above x_1 and x_4 will be omitted, and we write $\ln\sqrt{A} \equiv L$):

$$2R_{11} = -2[L_{11} - L_{44} - (x_1)^{-1}L_1] - (x_1)^{-2}(B_1^2 - C_1D_1) = 0; \quad (8)$$

$$2R_{44} = 2[L_{11} - L_{44} + (x_1)^{-1}L_1] - (x_1)^{-2}(B_4^2 - C_4D_4) = 0; \quad (9)$$

$$R_{14} = (x_1)^{-1}L_4 - (2x_1)^{-2}(2B_1B_4 - C_4D_1 - C_1D_4) = 0. \quad (10)$$

Half the sum of (8) and (9) gives

$$(x_1)^{-1}L_1 - (2x_1)^{-2}(B_1^2 + B_4^2 - C_1D_1 - C_4D_4) = 0. \quad (11)$$

Differentiating (10) with respect to x_1 and (11) with respect to x_4, it is easy to show, with the aid of (2), (3), and (4), that equations (10) and (11) are compatible, so that the difference in the expressions for L_{14} and L_{41} are proportional to

$$CD_4 + DC_4 - 2BB_4 = (CD - B^2)_4 = (x_1^2)_4 \equiv 0.$$

Finding the second derivatives L_{11} and L_{44} from (10) and (11), we readily see, with the aid of (2), (3), and (4), that equations (8) and (9) are also satisfied. This proves the compatibility of the whole system of equations.

Let us now return to the equations for B, C, and D. Eliminating B by means of $\alpha = x_1^2$ and introducing the new variables

$$\sigma \equiv C(CD - x_1^2)^{-1/2}, \quad \delta = D(CD - x_1^2)^{-1/2}, \quad (12)$$

we have

$$\begin{aligned} &[x_1(\sigma\delta - 1)^{-1/2}\sigma_1]_1 - x_1[(\sigma\delta - 1)^{-1/2}\sigma_4]_4 = 0, \\ &[x_1(\sigma\delta - 1)^{-1/2}\delta_1]_1 - x_1[(\sigma\delta - 1)^{-1/2}\delta_4]_4 = 0, \end{aligned} \quad (13)$$

corresponding to two non-linear interacting cylindrical waves. In the case studied by Einstein and Rosen[1] a linear cylindrical wave was obtained. A strong plane wave is impossible, because the metric curvature which it produces is incompatible with flat space (see Bergmann[2]).

Nevertheless it is a fact that, in spite of the non-linearity of the system of hyperbolic equations (13), their characteristics are the straight lines $x_1 = \pm x_4$. This shows that the strong gravitational waves propagate with a fundamental velocity equal to the speed of light, just as the weak ones do.* This is much more difficult to deduce from the general gravitational equations $R_{ik} = 0$. Notice further that the characteristics of each family obviously do not intersect. This means that the non-linear hyperbolic equations for gravity in free space do not necessarily lead to the formation of shock waves. In this they differ from the equations of gas dynamics, for example, which belong to the same class.

The space is Riemannian if it admits a continuous group of coordinate transformations. The occurrence of shock waves would contradict the basic hypothesis of continuous transformations; i.e., the Riemannian character of the metric. Thus our investigation has again shown the internal consistency of the Einstein equations for the gravitational field.

In conclusion I should like to thank V. L. Ginsberg for showing me the basic literature sources.

[1] A. Einstein and N. Rosen, J. Franklin Inst. **223**, 43 (1937).

[2] P. B. Bergmann, Introduction to the Theory of Relativity (New York, 1942). Russian translation, 1947, p 253.

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*It would nevertheless be interesting to study the general case $ds^2 = g_{ik}dx^idx^k$ where $g_{ik} = g_{ik}(x_1, x_4)$.