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A GROUP-THEORETICAL CONSIDERATION OF THE BASIS OF RELATIVISTIC QUANTUM MECHANICS. IV. SPACE REFLECTIONS IN QUANTUM THEORY

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The problem of invariance of relativistic quantum theory with respect to space reflections is investigated from the most general group-theoretical point of view, without the use of a specific form of the equations of motion. A complete classification is obtained for all the unitary and nonunitary irreducible representations of the improper inhomogeneous Lorentz group. In treating the problem of distinguishing representations according to parity, essential use is made of the concept of a universal covering group, which makes possible a rigorous proof of the previously mentioned[2] impossibility of simultaneous existence of spinor particles whose wave functions transform according to representations of different types.

An analysis is made of the experiment of Wu,[3] which is currently regarded as a proof of nonconservation of parity in weak interactions.[4] From the group-theoretical point of view developed here, experiments like those of Wu and Lederman[5] do not contradict parity conservation but merely conservation of charge conjugation, and consequently prove the electric charge is a pseudoscalar and that the electromagnetic potential is a pseudovector.

1. THE RELATION BETWEEN REPRESENTATIONS OF THE PROPER AND IMPROPER LORENTZ GROUPS

In the preceding papers I – III of this series,* we found all the irreducible representations of the inhomogeneous Lorentz group, i.e., all the possible laws of transformation under four-dimensional rotations and displacements for the wave functions of a relativistic quantum theory. The purpose of the present paper is to find all the irreducible representations of the improper group, which includes space reflections. The transition to the improper inhomogeneous Lorentz group G_s is accomplished by adjoining to the elements of the proper group G the inversion operation I_s, which satisfies the relations (I.34):

$$[I_s, p_0]_- = 0,\ [I_s, \mathbf{p}]_+ = 0,\ [I_s, \mathbf{M}]_- = 0,\ [I_s, \mathbf{N}]_+ = 0. \quad (1)$$

The first of these relations is a statement of the law of conservation of parity. Like the law of conservation of linear momentum, it is based on the fact that space and time variables are independent. According to (1), inversion changes the sign of the operators $\mathbf{p}$, $\mathbf{N}$ and leaves $\mathbf{M}$ and p_0 unchanged,

$$\mathbf{p} \to -\mathbf{p}, \quad p_0 \to p_0, \quad \mathbf{M} \to \mathbf{M}, \quad \mathbf{N} \to -\mathbf{N}. \quad (2)$$

It is easy to show that if the operators $\mathbf{p}$, p_0, $\mathbf{M}$, $\mathbf{N}$ form a representation P of the group G, the operators $-\mathbf{p}$, p_0, $\mathbf{M}$, $-\mathbf{N}$ also satisfy the commutation laws (I.33) and consequently also form a representation of the proper group, which we shall denote by I_sP.

If the representation I_sP is equivalent to P, then by definition there exists a non-degenerate matrix I_{s0} such that

$$I_{s0}^{-1}\mathbf{M}I_{s0} = \mathbf{M}, \quad I_{s0}^{-1}p_0I_{s0} = p_0,$$

$$I_{s0}^{-1}\mathbf{N}I_{s0} = -\mathbf{N}, \quad I_{s0}^{-1}\mathbf{p}I_{s0} = -\mathbf{p},$$

or,

$$[\mathbf{M}, I_{s0}]_- = 0,\ [p_0, I_{s0}]_- = 0,\ [\mathbf{N}, I_{s0}]_+ = 0,\ [\mathbf{p}, I_{s0}]_+ = 0. \quad (3)$$

Comparison of (1) and (3) shows that in this case the representation P is also a representation of the improper group G_s, in which the inversion operator is, except for a numerical factor λ_s, equal to the operator I_{s0} in (3),

$$P_s = P,\ I_s = \lambda_s I_{s0}. \quad (4)$$

But if the representation P is not equivalent to I_sP, then it can no longer be a representation of

*Notations introduced without explanation are the same as in the preceding papers of this series,[1] which are cited in the text as I, II, III. References like (I.33) refer to the corresponding formula in I.

the improper group G_s. In this case the representation P_s of the group G will be the direct sum of the representations P and I_sP:

$$P_s = P \dotplus I_s P. \tag{5}$$

The dimension of P_s is twice that of P, and the operators for the angular momentum $M^s_{\mu\nu}$, the momentum p^s_λ and the inversion I_s will have the form

$$\mathbf{M}^s = \begin{pmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M} \end{pmatrix}, \quad p_0^s = \begin{pmatrix} p_0 & 0 \\ 0 & p_0 \end{pmatrix}, \quad \mathbf{p}^s = \begin{pmatrix} \mathbf{p} & 0 \\ 0 & -\mathbf{p} \end{pmatrix},$$

$$\mathbf{N}^s = \begin{pmatrix} \mathbf{N} & 0 \\ 0 & -\mathbf{N} \end{pmatrix}; \tag{6}$$

$$I_s = \mu_s \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{7}$$

P_s is irreducible with respect to G_s if P is irreducible with respect to G and is not equivalent to I_sP. Thus, for each irreducible representation P of the proper group G, either P is equivalent to I_sP and constitutes an irreducible representation of the group G_s; or P is not equivalent to I_sP, and an irreducible representation of G_s is given by the direct sum $P \dotplus I_sP$. Later we shall show that the converse theorem is also true, i.e., any representation which is irreducible with respect to G_s is either irreducible with respect to G or is a direct sum of the type of (5) of two irreducible representations.

2. INVARIANTS OF THE IMPROPER INHOMOGENEOUS LORENTZ GROUP

From (I.34) and (I.42) it follows that I_s^2 and the fundamental invariants p_μ^2 and Γ_σ^2 of the improper group G_s. In II and III it was shown that for certain classes of representations there are additional invariants, which were discussed in Sec. 12 of I. These were the operator for the sign of the energy $S_H = p_0/|p_0|$ (II, Sec. 3), the operator for the sign of the fourth component of the intrinsic angular momentum $S_\Gamma = \Gamma_0/|\Gamma_0|$ (II, Sec. 4), the operator Σ of (III.16), (III.17), and finally the operators

$$F = {}^1/_2 M^2_{\mu\nu}, \quad W = (1/4\pi i)\,\varepsilon_{\mu\nu\lambda\sigma} M_{\mu\nu} M_{\lambda\sigma}$$

of (III.24), (III.25) for the class O_0. Of these operators, S_H, Σ, and F are scalars and commute with I_s, and consequently if they are invariants with respect to G, they are invariants with respect to G_s. The pseudoscalar operators S_Γ and W anticommute with I_s, and can be invariant relative to G without being invariant relative to G_s. However, if in a certain representation one of the pseudoscalar operators is invariant with respect to G, then its square is invariant with respect to G_s. We should also note that, according to the tables in II, III, the operators S_Γ, W are not simultaneously invariant in any of the irreducible representations of G.

3. STRUCTURE OF THE IRREDUCIBLE REPRESENTATIONS OF THE IMPROPER INHOMOGENEOUS LORENTZ GROUP

The irreducible representations of the proper group G, which were enumerated in II and III, can be divided into the following four types:

(1) $p_\mu \neq 0$, $W \neq \text{inv}$, $S_\Gamma \neq \text{inv}$,
(2) $p_\mu \neq 0$, $W \neq \text{inv}$, $S_\Gamma = \text{inv}$,
(3) $p_\mu = 0$, $W = \text{inv} \neq 0$,
(4) $p_\mu = 0$, $W = 0$.

In the first (fourth) case, there are no non-zero pseudoscalar operators in the representation which are invariant with respect to G and not with respect to G_s. In representations of this type, the operators $M_{\mu\nu}$, p_λ coincide with the corresponding operators in the representation of G, while the inversion operator has the form (4). In the second (third) case, the group contains one pseudoscalar operator $S_\Gamma(W)$, which is invariant with respect to G and not with respect to G_s. Then the operator $S_\Gamma^2(W^2)$ is invariant relative to G_s, and in an irreducible representation has a fixed numerical value $S^2_{\Gamma_0}(W_0^2)$. Thus a representation which is irreducible with respect to G_s can contain only two different irreducible representations of the group G, which differ in the values $\pm S_{\Gamma_0}(\pm W_0)$ of the pseudoscalar operator $S_\Gamma(W)$. The operator $S_\Gamma(W)$ changes sign under inversion, and the representation $P_\Gamma(P_W)$ of the group G changes into the representation $P_{-\Gamma}(P_{-W})$ which coincides with $I_sP_\Gamma(I_sP_W)$ to within an equivalence transformation I_{s0}. In accordance with (5) — (7), in this case the representation which is irreducible with respect to G_s is the direct sum

$$P_s = P_\Gamma \dotplus P_{-\Gamma} (P_s = P_{+W} \dotplus P_{-W}), \tag{8}$$

while the operators $M_{\mu\nu}$, p_λ, I_s will have the form

$$M_{\mu\nu} = \begin{pmatrix} M^\Gamma_{\mu\nu} & 0 \\ 0 & M^{-\Gamma}_{\mu\nu} \end{pmatrix}, \quad p_\mu = \begin{pmatrix} p^\Gamma_\mu & 0 \\ 0 & p^{-\Gamma}_\mu \end{pmatrix}, \tag{9}$$

$$I_s = \mu_s \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{10}$$

and correspondingly for P_W. In (10), μ_s is a number and I_{s0} is the equivalence transformation connecting $P_{-\Gamma}$ with I_sP_Γ (P_{-W} with I_sP_W).

We have thus proved the assertion made at the end of Sec. 1 that any irreducible representation of G_S is either irreducible with respect to G or is the direct sum of two irreducible representations.

Let us now construct explicitly the operator I_{S0} of (4) or (10). It is not difficult to see that for representations in the classes P_m, P_Π, P_0, this operator is the operator which changes the sign of the momentum:

$$I_{s0}\Omega(\mathbf{p}) = \Omega(-\mathbf{p}) \quad \text{for } P_m, P_\Pi, P_0. \tag{11}$$

In the class O_0, the basic functions of the irreducible representation $S^c_{k_0}$ have the form Ω^ν_k, where $\nu = -k, -k+1, \ldots, k$; $k = k_0, k_0+1, \ldots$ The angular momentum operator is diagonal with respect to k:

$$\mathbf{M}^{\nu\nu'}_{kk'} = \delta_{kk'}\mathbf{M}^{\nu\nu'}_{k}, \tag{12}$$

while the operator **N** has the form (cf., for example, Ref. 6)

$$\mathbf{N}^{\nu\nu'}_{kk'} = \mathbf{A}^{\nu\nu'}\delta_{kk'-1} + W\mathbf{B}^{\nu\nu'}\delta_{kk'} + \mathbf{C}^{\nu\nu'}\delta_{kk'+1}, \tag{13}$$

where $\mathbf{A}^{\nu\nu'}$, $\mathbf{B}^{\nu\nu'}$, $\mathbf{C}^{\nu\nu'}$ depend on F, W^2, k, ν, ν', (but not on W!). From (12) and (13) it follows that for $W = 0$ the operator

$$(I_{s0})_{kk'} = \delta_{kk'}(-1)^{k-k'} \tag{14}$$

commutes with **M** and anticommutes with **N**, i.e., can be used as the operator I_{S0} in (4). For $W \neq 0$, the operator (14) is also suitable for use as I_{S0}, but now of course, for (10). This is easily verified using (I.34), (12) and (13). In order to complete the construction of the irreducible representations of the group G_S, there remains for us only to determine the possible values of the factor λ_S in (4) and (10). However, for a rigorous treatment of this question, a more detailed investigation of the double-valued representations is necessary.

4. DOUBLE-VALUED REPRESENTATIONS AND UNIVERSAL COVERING GROUP

The double-valued representations are not representations in the strict sense of the word. It is, however, known that one can give a group for which these representations become single-valued (i.e., true) representations. This group is the universal covering group of the inhomogeneous Lorentz group. It is essential to note that the universal covering group is uniquely determined by the topological properties of the corresponding continuous group and is locally isomorphic to it (cf., for example, Ref. 7). The transition to the universal covering group is accomplished by adjoining to the transformations of the group G an element $I_{2\pi}$ of rotation through an angle 2π, which commutes with all the elements of the group and satisfies the relations

$$I^2_{2\pi} = \mathrm{I}, \ \lim I_\varphi = I_{2\pi} \ \text{for } \varphi \to 2\pi \tag{15}$$

for one of the spacial rotations. Since the spinor representations are true representations not of the Lorentz group G, but of its universal covering group which we shall denote by $\tilde{G}$, it is the latter group with respect to which the equations and wave functions of quantum theory are covariant. The element $I_{2\pi}$ is an invariant of the group $\tilde{G}$. It is equal to 1 for single-valued, and -1 for double-valued representations.*

In accordance with our previous remarks, the operation of inversion should be introduced into the group $\tilde{G}$. Twofold application of the inversion brings the system back into its original state, which can be interpreted as a rotation through either 0 or 2π. In the first case the square of the inversion operator is unity:

$$I^2_s = \mathrm{I}, \tag{16}$$

while in the second,

$$I^2_s = I_{2\pi}. \tag{17}$$

We emphasize that the relations (16), (17) do not define different representations of the same group, but rather different groups, which we denote by $\tilde{G}_S$ and $\tilde{G}'_S$ respectively. In other words, the relations (16), (17) give two types of structures of the space, and not of the individual particles.

5. THE SEPARATION OF REPRESENTATIONS OF THE GROUPS G_S, G'_S ACCORDING TO PARITY

In this section, we shall determine the possible values of the factors λ_S, μ_S in (4), (11) and thus complete the classification of the irreducible representations of the improper groups $\tilde{G}_S$, $\tilde{G}'_S$. For single-valued (unprimed) representations P, two-fold application of the inversion gives the identity

$$I^2_s = \lambda^2_s = \mathrm{I}. \tag{18}$$

so that

$$\lambda_s = \pm 1. \tag{19}$$

*We shall continue to use the usual terms "single-valued" and "double-valued" representations, even though all representations are single-valued with respect to $\tilde{G}$.

The representations corresponding to $\lambda_S = 1$ and $\lambda_S = -1$ are not equivalent to one another if (4) holds for them. They are said to be even and odd, respectively, and will be denoted by $+P$ and $-P$. The occurrence of the parity assignment doubles the number of possible single-valued irreducible representations of each of the groups $\tilde{G}_S$, $\tilde{G}'_S$ for the case when (4) applies. This is the case for the representations in the classes

$$P^s_{\pm m},\ P^a_{\text{II}},\ P^s_{\text{II}},\ P^b_{\text{II}},\ P^\alpha_{\text{II}},\ P^c_{\pm 0},\ P^{-c}_{\pm 0},\ P^\alpha_{\pm 0}$$

and those representations of the class O_0 for which $W = 0$ (this set includes, in particular, four-dimensional vectors and tensors).

The double-valued representations of the groups $\tilde{G}_S$ and $\tilde{G}'_S$ differ from one another. For the double-valued representations of $\tilde{G}_S$,

$$I_s^2 = \lambda_s^2 = 1,\ \lambda_s = \pm 1, \tag{20}$$

while for $\tilde{G}'_S$,

$$I_s^2 = \lambda_s^2 = I_{2\pi} = -1,\ \lambda = \pm i \tag{21}$$

The double-valued representations of the group $\tilde{G}_S$ with $\lambda_S = 1$ and $\lambda_S = -1$ have opposite parity and are not equivalent to one another. For example, the direct product of two representations with $\lambda_S = 1$ is different from the product of a representation with $\lambda_S = 1$ and a representation with $\lambda_S = -1$. At the same time, a well-defined parity cannot be assigned to either of the representations, since for each of them there exist two operators I_S and $I_S I_{2\pi}$, one of which multiplies the wave function by 1, the other by -1, and each of them has an equal right to be regarded as the inversion operator. Therefore, for the two-valued representations we can speak only of mutual or relative parity. As pointed out in Ref. 8, this situation was first noted by Landau. An analogous situation occurs for the double-valued representations of the group $\tilde{G}'_S$ with $\lambda_S = i$ and $\lambda_S = -i$, which we shall denote by $+iP'$ and $-iP'$, respectively. Since the representations $\pm P'$ and $\pm iP'$ refer to different groups, it is meaningless to talk of direct products of the type $\pm P' \times \pm iP'$. This means that in real space only those physical systems can occur whose wave functions transform according to double-valued representations of the type $\pm P'$ alone, or of the type $\pm iP'$ alone. Arguments concerning the impossibility of simultaneous existence of spinors which are multiplied by ± 1 and by $\pm i$ under inversion have been given previously by Shapiro.[2] The use of the concept of the universal covering group enables us to formulate these arguments as a rigorous proof. Another difference between our treatment and that of, say, Shapiro, is that in this paper we investigate the irreducible representations of the group G_S rather than the Dirac equation. Thus the results obtained are applicable to particles of arbitrary spin.

We emphasize that a distinction according to parity exists only for representations which satisfy (4). When condition (10) is satisfied, the concept of parity of the representation cannot be introduced, since the representations corresponding to $\mu_S = 1$ and $\mu_S = -1$ are related by the equivalence transformation $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. This is the situation for the representations

$$P^\Sigma_{\pm 0} = P^{+\Sigma}_{\pm 0} + P^{-\Sigma}_{\pm 0},\ P^l_{\text{II}} = P^{+l}_{\text{II}} + P^{-l}_{\text{II}} \tag{22}$$

and those representations of the class O_0 which have $W \neq 0$ (and which include the Dirac bispinor as a special case).

For completeness of our discussion, we remark that in both $\tilde{G}_S$ and $\tilde{G}'_S$ there is still one nontrivial one-dimensional irreducible representation, the representation of the factor group of G_S with respect to the subgroup G. For this representation, which we denote by J_S:

$$M_{\mu\nu} = 0,\ p_\lambda = 0,\ I_s = -1.$$

It is easily shown that

$$-P = J_s \times (+P),\ -P' = J_s \times (+P'),$$
$$-iP' = J_s \times (+iP'). \tag{23}$$

6. INTERNAL PARITY OF ELEMENTARY PARTICLES

So far, we have treated parity defined as the eigenvalue of the operator of inversion with respect to the coordinate origin. Such a definition is used in treating collisions and other similar processes, where the origin is chosen to be the center of inertia of the physical system. In addition to this, we can define for each individual particle an internal parity which is an invariant characteristic of the particle. The internal parity λ_S of a particle can be introduced by means of the relation

$$\lambda_s = I_s I_{s0}, \tag{24}$$

where I_S is the usual (external) parity, and I_{S0} is the operator for the Lorentz transformation which takes the 4-momentum $(\mathbf{p}, ip_0)$ into $(-\mathbf{p}, ip_0)$. For $\mathbf{p} = 0$, $I_{S0} = 1$ and the external parity coincides with the internal. Obviously if I_{S0} in (4) is chosen in the form of (11), the factor λ_S in (4) will determine the internal parity. The concept of internal parity exists only for particles with non-zero rest mass, which are described by the representations P_m^s, $P_m'^s$ satisfying (4). Particles with zero rest mass do not possess an internal parity, since their wave functions transform according to the representations P_{+0}^{Σ}, $P_{+0}'^{\Sigma}$, which satisfy (10) and not (4). Thus, for example, it is meaningless to speak of a photon as being a vector or pseudovector particle.

7. CONSERVATION OF PARITY IN WEAK INTERACTIONS

In pseudo-Euclidean space-time, the inversion and time displacement transformations commute, so that it automatically follows that the inversion operator I_s commutes with the Hamiltonian, i.e., I_s is conserved. Consequently, since the experiments of Wu,[3] Lederman,[5] and others do not contradict the pseudo-Euclidean character of space-time,[4,9] they cannot contradict the law of conservation of parity, but merely show that the present definition of parity is incorrect. In fact, by definition parity is the eigenvalue of the inversion operator, which is used to transform the wave function when all three space coordinates are reflected. The inversion operator must therefore necessarily satisfy the commutation relations (1), which are simply a mathematical statement of the definition of parity.

For example, let us consider from this point of view the experiments on the β-decay of polarized nuclei which were proposed by Lee and Yang[4] and carried out by Wu.[3] In these experiments, the angular distribution of electrons emitted by Co^{60} nuclei polarized in a magnetic field was studied. An asymmetry of the cross section with respect to the plane perpendicular to the field was observed. It follows that the angular distribution must contain a term proportional to the scalar product of the magnetic field **H** and the electron momentum **p**,

$$\sigma \sim a + b\mathbf{p}\cdot\mathbf{H}. \tag{25}$$

We emphasize that the quantity which is directly observable is $\mathbf{p}\cdot\mathbf{H}$ and not $\mathbf{p}\cdot\mathbf{s}$, where **s** is the spin of the nucleus. Since the cross section is a scalar, the quantity $\mathbf{p}\cdot\mathbf{H}$ is also a scalar, but since according to (1) **p** is a vector, the magnetic field **H** must also be a vector. The results of Wu's experiment also essentially reduce to this assertion. Up to now the magnetic field was considered to be a pseudovector and not a vector. However, until the experiment of Wu there was no possibility of establishing the law of transformation of the electromagnetic field under inversion. In fact, in the Maxwell equations and the expression for the Lorentz force density f_μ:

$$\Box A_\mu = -4\pi j_\mu, \quad f_\mu = F_{\mu\nu} j_\nu$$

A_μ and j_μ can be regarded as being either vectors or pseudovectors. If we take the second point of view, charge should be treated as a pseudoscalar, the electric field as a pseudovector, etc. The question whether A_μ and j_μ are actually vectors or pseudovectors cannot be answered within the framework of electrodynamics, since it reduces to equations which are satisfied in both cases. The two possible laws of reflection are different for an operation which consists in the simultaneous change of sign of A_μ and j_μ, i.e., for charge conjugation. Thus the occurrence of two possible laws for inversion in electrodynamics is related to its invariance under charge conjugation. These statements remain valid in a quantum theory. The Wu experiment shows that weak interactions are not invariant with respect to charge conjugation, and that in electrodynamics the second of the possible choices for the reflection operation, which has been called combined inversion,[9] is the correct one, and that, in particular, charge is a pseudoscalar.

Since charge is a pseudoscalar, it anticommutes with the inversion. Therefore for charged particles the parity cannot have a definite value. However, because of the invariance of strong interactions with respect to charge conjugation, the product of true inversion and charge conjugation is an approximate integral of the motion of such particles. This operator commutes with the charge, and its eigenvalues, which are incorrectly called the parity, can have definite values for charged particles also.

To a considerable extent, these considerations have a methodological character. However, the bringing of clarity into the definition of the concept of parity is necessary at the present time because of the widespread use of the, from our point of view, extremely misleading term "nonconservation of parity in weak interactions," when we are actually dealing with the nonconservation of charge conjugation and the problem of correct choice of the inversion operation for specific equations of motion.

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