small (< 1 Bev), then, as a consequence of the considerable divergence angle of the electron and the positron, the length of the track is, as a rule, too short for the application of method 2. On the other hand, in the most cases, at electron energies > 30 Bev, measurement of the Coulomb scattering is impossible. Exceptions are pairs with tracks ≥ 4 cm long within the limits of the emulsion.

It is advisable to employ method 1 at energies < 1 Bev and method 3 at energies > 30 Bev.

A trident was formed 19,355μ from the start of pair No. 1. True tridents satisfy the following criterion[10]

$$\theta_i > \bar{\alpha}_i \quad i = 1,\ 2,\ 3,$$

where θ_i is the angle that the i-th electron track of the trident makes with the continuation of the primary track; $\bar{\alpha}_i$ is the average angle of multiple scattering (including the Coulomb and all other types of scattering) of the i-th electron. It follows from Table V that this criterion is satisfied here.

On the other hand, according to the plot obtained by Kaplan and Koshiba,[10] the probability that this is a bremsstrahlung pair, less than 0.2μ away from the primary electron track, is nearly 8%. These circumstances make it possible to assert that the trident is a true one. The total length of all electron tracks in the reviewed band is 25 cm. Consequently, the average length of formation of the trident at an energy of 6 Bev is on the order of 25 cm, which is in agreement with the Bhabha theory.[14]

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GENERAL COVARIANT EQUATIONS FOR FIELDS OF ARBITRARY SPIN

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The Gel'fand-Iaglom field equations are extended to the general theory of relativity.

To obtain a generalized wave equation for a field in general covariant form, one must define covariant differentiation of a generalized wave function describing particles with arbitrary spin. Gel'fand and Iaglom,[1] Dirac,[2] and Fierz and Pauli[3] have studied the generalized wave equation in the special theory of relativity. In the present article, their theory is extended to the general covariant form.

1. SEMIMETRICS AND SEMIMETRIC REPRESENTATION

We introduce the metric g_{ik} in space-time with the aid of the asymmetric matrix $\| \lambda_{i(\alpha)} \|$

according to[4]

$$g_{ik} = \lambda_{i(\alpha)}\lambda_{k(\alpha)}, \quad \|\lambda_{i(\alpha)}\|^2 = \|g_{ik}\|. \tag{1}$$

Formally $\lambda_{i(\alpha)}$ may be thought of as half the metric g_{ik}. We shall call it the semimetric, and the representation shall be called the semimetric representation. The metric g_{ik} remains invariant if the semimetric $\lambda_{i(\alpha)}$ is subjected to the orthogonal transformation

$$\lambda'_{i(\alpha)} = L_{(\alpha\beta)}\lambda_{i(\beta)}, \tag{2}$$

where $\|L_{(\alpha\beta)}\|$ is an orthogonal matrix, which means that

$$L_{(\alpha\beta)}L_{(\alpha\gamma)} = \delta_{(\beta\gamma)}. \tag{3}$$

This is easily seen from (2) and (3), according to which

$$g'_{ik} = \lambda'_{i(\alpha)}\lambda'_{k(\alpha)} = \lambda_{i(\alpha)}\lambda_{k(\alpha)} = g_{ik}. \tag{4}$$

Therefore all the equations of the general theory of relativity must remain invariant with respect to two transformation groups, namely, (a) the group of general transformations of all coordinates of the form

$$x'^i = f^i(x^1, x^2, x^3, x^4), \tag{5}$$

and (b) the group of orthogonal transformations of the elements of the semimetric matrix

$$\lambda'_{i(\alpha)} = L_{(\alpha,\beta)}\lambda_{i(\beta)}. \tag{6}$$

According to Eq. (1)

$$|\operatorname{Det}\lambda_{i(\alpha)}| = +\sqrt{\operatorname{Det} g_{ik}} \neq 0. \tag{7}$$

Denoting the elements of the inverse matrix $\|\lambda_{i(\alpha)}\|^{-1}$ by $\lambda^i_{(\alpha)}$, we have

$$\lambda^i_{(\alpha)}\lambda_{i(\beta)} = \delta_{(\alpha\beta)}; \quad \lambda^i_{(\alpha)}\lambda_{k(\alpha)} = \delta^i_k. \tag{8}$$

We define

$$dx_{(\alpha)} = \lambda_{i(\alpha)}\,dx^i, \quad dx^i = \lambda^i_{(\alpha)}\,dx_{(\alpha)}. \tag{9}$$

We shall call these new coordinates $x_{(\alpha)}$ the semimetric coordinates. From (8) and (1) it follows that

$$ds^2 = g_{ik}\,dx^i\,dx^k = dx_{(\alpha)}\,dx_{(\alpha)}. \tag{10}$$

This means that in semimetric space the element of length is given by a normal quadratic form and is invariant under the linear transformation

$$x'_{(\alpha)} = L_{(\alpha\beta)}x_{(\beta)}. \tag{11}$$

From (9) we obtain

$$\lambda_{i(\alpha)} = \partial x_{(\alpha)}/\partial x^i, \quad \lambda^i_{(\alpha)} = \partial x^i/\partial x_{(\alpha)}. \tag{12}$$

It follows from (11) that

$$\frac{\partial\varphi}{\partial x_{(\alpha)}} = \frac{\partial\varphi}{\partial x^i}\frac{\partial x^i}{\partial x_{(\alpha)}} = \lambda^i_{(\alpha)}\frac{\partial\varphi}{\partial x^i},$$

so that

$$\lambda^i_{(\alpha)}\,\partial/\partial x^i \to \partial/\partial x_{(\alpha)}. \tag{13}$$

2. COVARIANT DERIVATIVE OF A GENERALIZED FIELD FUNCTION

Let us introduce an n-dimensional matrix vector in semimetric space, whose components $L_{(\alpha)}$ form a set of n Hermitian matrices satisfying the condition

$$[L_{(\alpha)}, I_{(\beta\gamma)}]_- = \delta_{(\alpha\beta)}L_{(\gamma)} - \delta_{(\alpha\gamma)}L_{(\beta)}, \tag{14}$$

where $I_{(\alpha\beta)}$ is an infinitesimal operator of a representation of the group of linear transformations of Eq. (11). These infinitesimal operators satisfy the commutation rules

$$[I_{(\alpha\beta)}, I_{(\gamma\delta)}]_- = \delta_{(\alpha\gamma)}I_{(\beta\delta)} + \delta_{(\beta\delta)}I_{(\alpha\gamma)} - \delta_{(\alpha\delta)}I_{(\beta\gamma)} - \delta_{(\beta\gamma)}I_{(\alpha\delta)}. \tag{15}$$

We shall denote the contravariant and covariant components of this matrix vector in Riemannian space by

$$L^i = \lambda^i_{(\alpha)}L_{(\alpha)}; \quad L_i = \lambda_{i(\alpha)}L_{(\alpha)}. \tag{16}$$

Writing

$$I_{ij} = \lambda_{i(\beta)}\lambda_{j(\gamma)}I_{(\beta\gamma)},$$

we can readily obtain from (14), (15), and (1) the relations

$$[L_i, I_{jk}]_- = g_{ij}L_k - g_{ik}L_j; \tag{17}$$

$$[I_{ij}, I_{kl}]_- = g_{ik}I_{jl} + g_{jl}I_{ik} - g_{il}I_{jk} - g_{jk}I_{il}. \tag{18}$$

Two complex generalized field functions ψ and $\bar\psi$ are called adjoint functions if the n Hermitian forms

$$\bar\psi L_{(\alpha)}\psi$$

make up a vector in semimetric space. Under transformations of group (a)

$$\bar\psi' L_{(\alpha)}\psi' = \bar\psi L_{(\alpha)}\psi, \tag{19}$$

the components of these vectors remain invariant. Under the transformations of group (b), we have

$$\bar\psi' L_{(\alpha)}\psi' = L_{(\alpha\beta)}(\bar\psi L_{(\beta)}\psi). \tag{20}$$

The generalized wave functions transform among themselves according to

$$\psi' = S\psi, \quad \bar\psi' = \bar\psi S^{-1}, \tag{21}$$

where S is a matrix which varies from point to point and is related to $\|L_{(\alpha\beta)}\|$ by

$$SL_{(\alpha)}S^{-1} = L_{(\alpha\beta)}L_{(\beta)}, \tag{22}$$

which follows from (20). In order to derive the covariant differentiation formula for a generalized field function, we must define the concept of parallel displacement. If two points x and x + dx are separated by an infinitesimal distance, the wave functions at these points are related by the infinitesimal linear transformations

$$\psi(x+dx) = [I + \Lambda_i\, dx^i]\,\psi(x),$$
$$\bar{\psi}(x+dx) = \bar{\psi}(x)\,[I - \Lambda_i\, dx^i], \tag{23}$$

where Λ is a certain matrix.

If (23) is to define parallel displacement, the vector $\bar{\psi}L_{(\alpha)}\psi$, which is constructed of $\bar{\psi}$ and ψ, must undergo a parallel displacement, which means that

$$A_{(\alpha)}(x+dx) = \{\delta_{(\alpha\beta)} + \eta_{\sigma(\alpha\beta)}\, dx^\sigma\}\, A_{(\beta)}(x). \tag{24}$$

From (23) and (24) it follows that

$$\bar{\psi}(x)\,[I - \Lambda_i\, dx^i]\, L_{(\alpha)}\,[I + \Lambda_i\, dx^i]\,\psi(x)$$
$$= \bar{\psi}L_{(\beta)}\,\psi\,[\delta_{(\alpha\beta)} + \eta_{i\,(\alpha\beta)}\, dx^i], \tag{25}$$

where the $\eta_{i(\alpha\beta)}$ are the components of the affine connection, which have been given by Rumer.[4]

According to Eq. (25), the Λ_i are given by

$$L_{(\alpha)}\Lambda_i - \Lambda_i L_{(\alpha)} = \eta_{i(\alpha\beta)}\, L_{(\beta)}. \tag{26}$$

Multiplying (26) by $\lambda^i_{(\delta)}$, we obtain

$$L_{(\alpha)}\Lambda_{(\delta)} - \Lambda_{(\delta)}L_{(\alpha)} = \eta_{(\delta\alpha\beta)}L_{(\beta)}, \text{where}\, \Lambda_{(\delta)} = \lambda^i_{(\delta)}\Lambda_i; \tag{27}$$

where the $\eta_{(\alpha\beta\gamma)} = \lambda^i_{(\alpha)}\eta_{i(\beta\gamma)}$ are the Ricci curvature coefficients.[5]

The general solution of Eq. (27) will be

$$\Lambda_i = {}^1\!/_2\, \eta_{i(\alpha\beta)}\, I_{(\alpha\beta)} + i f_i I, \tag{28}$$

where the f_i are arbitrary functions. This is easily seen by making use of (14).

Writing $f_{(\alpha)} = \lambda^i_{(\alpha)} f_i$, we obtain

$$\Lambda_{(\delta)} = {}^1\!/_2\, \eta_{(\delta\alpha\beta)} I_{(\alpha\beta)} + i f_{(\delta)} I. \tag{29}$$

Thus the covariant derivative of a generalized field function will be

$$\nabla_i\psi = \partial\psi/\partial x^i - \Lambda_i\psi, \quad \nabla_i\bar{\psi} = \partial\bar{\psi}/\partial x^i + \bar{\psi}\Lambda_i \tag{30}$$

and in semimetric space

$$\nabla_{(\alpha)}\psi = \partial\psi/\partial x_{(\alpha)} - \Lambda_{(\alpha)}\psi, \quad \nabla_{(\alpha)}\bar{\psi} = \partial\bar{\psi}/\partial x_{(\alpha)} + \bar{\psi}\Lambda_{(\alpha)}. \tag{31}$$

3. EXTENSION OF THE GEL'FAND-IAGLOM FIELD EQUATIONS TO THE GENERAL THEORY OF RELATIVITY

The covariant field equations for arbitrary spin were obtained in the special theory of relativity by Gel'fand and Iaglom,[1] and are of the form

$$L^k\,\partial\psi/\partial x^k + m\psi = 0, \tag{32}$$

where ψ is a generalized field function describing particles with arbitrary spin, and the L^k are matrices which determine the linear transformation properties of the ψ function.

In order that Eqs. (32) become covariant with respect to all physically possible transformations, the ordinary derivatives $\partial\psi/\partial x^k$ which appear in them must be replaced by covariant derivatives $\nabla_k\psi$. Then the general covariant field equations will be

$$L^k\nabla_k\psi + m\psi = 0. \tag{33}$$

Inserting (30) into (33) and using (28), we obtain

$$L^k\,\partial\psi/\partial x^k + m\psi - {}^1\!/_2\, L^k\eta_{k(\beta\gamma)}\, I_{(\beta\gamma)}\psi = 0, \tag{34}$$

where the L^k are matrix functions satisfying Eqs. (17). From (13), (16), and (34) one can obtain the general covariant field equations in semimetric space, namely

$$L_{(\alpha)}\,\partial\psi/\partial x_{(\alpha)} + m\psi - {}^1\!/_2\,\eta_{(\alpha\beta\gamma)}L_{(\alpha)}I_{(\beta\gamma)}\psi = 0, \tag{35}$$

where the $L_{(\alpha)}$ satisfy relations (14). If $L_{(\alpha)} = \gamma_{(\alpha)}$, where the $\gamma_{(\alpha)}$ are Dirac matrices, then

$$I_{(\alpha\beta)} = {}^1\!/_2\,[\gamma_{(\alpha)}\gamma_{(\beta)} - \gamma_{(\beta)}\gamma_{(\alpha)}], \tag{36}$$

and (34) become the general covariant Dirac equation

$$\gamma_{(\alpha)}\,\partial\psi/\partial x_{(\alpha)} + m\psi - {}^1\!/_4\,\gamma_{(\alpha)}\gamma_{(\beta)}\gamma_{(\gamma)}\eta_{(\alpha\beta\gamma)}\psi = 0, \tag{37}$$

a special case of (34) which has been previously obtained by Fock and Ivanenko.[6] This shows that Eqs. (34) and (35) are of greater generality. The general covariant Lagrangian is in this case

$$L = {}^1\!/_2\left\{\bar{\psi}L^k\left(\frac{\partial\psi}{\partial x^k} - \Lambda_k\psi\right) - \left(\frac{\partial\bar{\psi}}{\partial x^k} + \Lambda_k\bar{\psi}\right)L^k\psi + 2m\bar{\psi}\psi\right\}. \tag{38}$$

If (38) is substituted into Euler's equation, one easily obtains Eq. (34). Further, it is easily shown that the symmetric energy-momentum tensor and the current vector in general covariant form can be written, respectively,

$$T_{ik} = {}^1\!/_2\,\bar{\psi}L_k\nabla_i\psi + \bar{\psi}L_i\nabla_k\psi - \nabla_i\bar{\psi}L_k\psi - \nabla_k\bar{\psi}L_i\psi], \tag{39}$$

$$j^k = ie\bar{\psi}L^k\psi. \tag{40}$$

It should be noted that the field equations (35), as opposed to those of the special theory of relativity, contain the additional operator terms

$$\delta\hat{m} = {}^1\!/_2\,\eta_{(\alpha\beta\gamma)}L_{(\alpha)}I_{(\beta\gamma)},$$

acting on the spinor or tensor fields. These operators may therefore be treated as mass operators entering the theory in a natural way. They are not introduced artificially or without any reasonable basis, as is done in many works on ordinary quantum field theory. One may hope that these operators will make it possible to eliminate the difficulties associated with divergences in field theory.

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CIRCULAR WAVES IN AN ELECTRON-ION BEAM

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We consider circular waves in an uncompensated electron-ion beam in a cylindrical waveguide with perfectly conducting walls. The magnetic field produced by the beam current is assumed to be very strong. We treat the problem qualitatively, without an exact solution of the differential equations. It is shown that the beam is stable with respect to the oscillations being considered, and the natural frequency bands are found. The electromagnetic field is mapped.

1. The stationary state of an electron-ion beam has been examined by Bennett[1] and Budker.[2]

In the present article we consider the oscillations of an electron-ion beam in a cylindrical waveguide of radius R. We shall treat circular oscillations, which means that we assume the electromagnetic field, as well as the electron and ion densities and velocities, to be independent of z, and the dependence of these quantities on r, φ, and the time t to be given by

$$F(r, \varphi, t) = f(r)\, e^{i(\omega t - \mu\varphi)}. \qquad (1)$$

The conclusions we reach can be extended to perturbations of a more general form, namely

$$F(r, \varphi, z, t) = f(r)\, e^{i(\omega t - \mu\varphi - \gamma z)},$$

so long as the condition $\gamma R \ll 1$ is satisfied.

The amplitude of vibrations is considered small, and the equations are linearized. The problem is solved in the hydrodynamic approximation; the electrons and ions have different temperatures, which are constant in space and time.

We assume also that the magnetic field produced by the beam current is so strong that the inequalities

$$|eH_{\varphi 0}(r)/mc\omega| \gg 1, \quad |eH_{\varphi 0}(r)/Mc\omega| \gg 1 \qquad (2)$$

are fulfilled at all points within the waveguide. Here $H_{\varphi 0}(r)$ is the magnetic field strength, ω is the frequency

$$m = m_0/\sqrt{1-\beta_e^2}, \; M = M_0/\sqrt{1-\beta_i^2}; \; \beta_e = v_{0e}/c, \; \beta_i = v_{0i}/c,$$

m_0 and M_0 are the electron and ion rest masses, and v_{0e} and v_{0i} are the electron and ion velocities in the stationary state. Since we are dealing with a strong magnetic field, the variable components v_{er}, v_{ez}, v_{ir}, and v_{iz} of the electron and ion velocities are small.

We shall show below that the beam is stable with respect to the oscillations we are considering, and