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# ON THE THEORY OF FERROMAGNETIC SUPERCONDUCTORS

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We obtain the conditions for the existence of a superconducting state in bulk, single-domain ferromagnetic samples in the shape of an ellipsoid of revolution. The direction of the spontaneous magnetization is assumed to make an arbitrary angle $\theta_0$ with the direction of the external field, which is parallel to the axis of the ellipsoid. We give estimates from which it follows that the superconducting state can occur only if the angle $\theta_0 \lesssim 10^{-2}$. If this condition is satisfied, the use of oblate samples with a large demagnetization factor should formally favor the possibility of detecting the superconductivity.

1. The problem of a possible observation of superconductivity in ferromagnetics was considered in a paper by Ginzburg.[1] It was shown that in the case of a bulk cylindrical specimen, magnetized along the axis of the cylinder, the possibility of a transition to a superconducting state was hampered because of the presence inside the specimen of the magnetic induction $B_0 = 4\pi M_0$, which is connected with the spontaneous magnetization $M_0$ of the ferromagnetic. For specimens with a large demagnetization factor (for instance, for thin disks magnetized perpendicularly to their plane) the induction B in the sample in the normal state turns out to be very small compared to $4\pi M_0$ which can assist the onset of superconductivity. Ginzburg, when considering this, pointed out at the same time the necessity of a special analysis of that case, taking boundary effects into account. In the present paper we elucidate the conditions for the existence of a superconducting phase in ferromagnetics of finite dimensions and ellipsoidal shape with a spontaneous magnetization at an arbitrary angle with the direction of the external field.

2. It is well-known[2,3] that if the processes considered take place at constant temperature T and everywhere constant and uniform magnetic field $H_0$, the following function is extremal in the equilibrium state,

$$\Phi(T, H_0) = \int F_H\, dv - \frac{1}{4\pi}\int \mathbf{H}_0\mathbf{B}\, dv + \frac{1}{8\pi}\int H_0^2\, dv. \tag{1}$$

Here $F_H$ is the internal free energy density of the system under consideration, taking the total magnetic field H into account, B is the induction field, and the integration is over the whole of space.

For a solid in the normal state the free energy is of the form

$$\int F_{nH}\, dv = \int F_{n0}\, dv + \int w_m\, dv, \tag{2}$$

where $F_{n0}$ is the free energy density when no magnetic field is present and $w_m$ the free energy density connected with the field. For a ferromagnetic we must take

$$w_m = \frac{1}{4\pi}\int_0^B \mathbf{H}d\mathbf{B} = \frac{H^2}{8\pi} - 2\pi M_0^2 = \frac{B^2}{8\pi} - \mathbf{M}_0\mathbf{B}, \tag{3}$$

where we have used in (3) the expression $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}_0$ for the induction inside an "ideal" ferromagnetic (the magnetic permeability $\mu$ is put equal to unity, which corresponds to the saturated case). We shall assume that the specimen consists of one domain and that at liquid helium temperatures $\mathbf{M}_0$ is constant and does not depend on the temperature.

From (1) to (3) we get for a specimen in the normal state

$$\Phi_n(T, H_0) = \int F_{n0}\, dv^- + \frac{1}{8\pi}\int (\mathbf{H}-\mathbf{H}_0)^2\, dv - \mathbf{M}_0\mathbf{H}_0 v^- - 2\pi M_0^2 v^-, \tag{4}$$

where $v^-$ is the volume of the specimen.

We shall consider specimens which are sufficiently large so that we can neglect surface effects. For a ferromagnetic in the superconducting state the free energy is equal to

$$\int F_{sH}\, dv = \int F_{s0}\, dv + \int\left(\frac{B^2}{8\pi} - \mathbf{M}_0\mathbf{B}\right) dv = \int F_{s0}\, dv + \frac{1}{8\pi}\int (H^+)^2\, dv^+,$$

inasmuch as within a superconductor B = 0 and outside the specimen $B = H^+$ ($H^+$ is the field outside the specimen and $v^+$ the volume outside the specimen). For the thermodynamic potential (1) we get

$$\Phi_s(T, H_0) = \int F_{s0}\, dv^- + \frac{1}{8\pi}\int H_0^2\, dv^- + \frac{1}{8\pi}\int (\mathbf{H}^+ - \mathbf{H}_0)^2\, dv^+. \tag{5}$$

Equilibrium between the normal and the superconducting phases is possible only if the potentials are equal,

$$\Phi_n(T, H_0) = \Phi_s(T, H_0). \tag{6}$$

Below we evaluate the quantities (4) and (5) for the case of samples having the shape of ellipsoids of revolution with the axis of revolution parallel to the external field.

**3.** First we find the magnetic field outside and inside a prolate ellipsoid of revolution, the spontaneous magnetization of which is in the direction which makes an angle $\theta_0$ with the direction of the external field. Introducing spheroidal coordinates $\xi, \eta, \phi$ (see, for instance, Ref. 4) and solving the usual magnetostatic problem in the case of a specimen in the normal state, we get for the scalar potential of the magnetic field the expression

$$\varphi^+ = c\xi\eta\,[H_0 + A_1 I_1(\xi)] + c\sqrt{\xi^2-1}\sqrt{1-\eta^2}A_2 I_2(\xi)\cos\phi,$$
$$\varphi^- = c\xi\eta\,[H_0 + A_1 I_1(\xi_0)] + c\sqrt{\xi^2-1}\sqrt{1-\eta^2}A_2 I_2(\xi_0)\cos\phi; \tag{7}$$

$$A_1 = -4\pi M_0\xi_0(\xi_0^2-1)\cos\theta_0,\quad A_2 = 4\pi M_0\xi_0(\xi_0^2-1)\sin\theta_0,$$
$$I_1(\xi) = \int_\xi^\infty d\xi/\xi^2(\xi^2-1),\quad I_2(\xi) = \int_\xi^\infty d\xi/(\xi^2-1)^2. \tag{7a}$$

The coordinate surface $\xi = \xi_0$ coincides with the surface of the ellipsoid considered, the equation of which is of the form

$$(x^2+y^2)/(\xi_0^2-1) + z^2/\xi_0^2 = c^2,$$

where 2c is the interfocal distance.

In the case of a superconducting ellipsoid (inside the specimen $B^- = 0$) we find

$$\varphi^+ = c\xi\eta H_0 + \frac{H_0\xi_0(\xi_0^2-1)}{1 - I_1(\xi_0)\xi_0(\xi_0^2-1)}c\xi\eta I_1(\xi). \tag{8}$$

For the case of an oblate ellipsoid of revolution we obtain similar expressions.

Evaluating the field $\mathbf{H} = -\operatorname{grad}\varphi$ and performing the integrations indicated in (4) and (5), we obtain equations which are correct both for a prolate and for an oblate ellipsoid of revolution.*

$$\Phi_n = \int F_{n0}\, dv - \frac{v^-}{8\pi}(4\pi M_0)^2[(1-n_1)\cos^2\theta_0 + (1-n_2)\sin^2\theta_0] - v^- M_0 H_0\cos\theta_0,$$
$$\Phi_s = \int F_{s0}\, dv + \frac{v^-}{8\pi}\frac{H_0^2}{1-n_1}. \tag{9}$$

In the case of a prolate ellipsoid of revolution the quantities $n_1$ and $n_2$ entering into (9) have the form

$$n_1 = \xi_0(\xi_0^2-1)\left[\frac{1}{2}\ln\frac{\xi_0+1}{\xi_0-1} - \frac{1}{\xi_0}\right],$$
$$n_2 = \xi_0(\xi_0^2-1)\left[\frac{\xi_0}{2(\xi_0^2-1)} - \frac{1}{4}\ln\frac{\xi_0+1}{\xi_0-1}\right]. \tag{10}$$

In the case of an oblate ellipsoid of revolution, the equation of which is

$$(x^2+y^2)/\xi_0^2 + z^2/(\xi_0^2-1) = c^2,$$

the quantities $n_1$ and $n_2$ have the form ($\rho_0 = \sqrt{\xi_0^2 - 1}$),

$$n_1 = \rho_0(\rho_0^2+1)\left[\tan^{-1}\rho_0 + \frac{1}{\rho_0} - \frac{\pi}{2}\right],$$
$$n_2 = \rho_0(\rho_0^2+1)\left[\frac{\pi}{4} - \frac{\rho_0}{2(\rho_0^2+1)} - \frac{1}{2}\tan^{-1}\rho_0\right]. \tag{11}$$

**4.** Equating the thermodynamic potentials (9) we find the critical external magnetic field $H_{cr}$ for which the normal and the superconducting phase can exist in equilibrium with one another,

$$H_{cr} = -4\pi M_0\cos\theta_0 \pm R;$$
$$R = \{8\pi\Delta(1-n_1) - (4\pi M_0)^2(1-n_1)(1-n_2)\sin^2\theta_0\}^{1/2},$$
$$\Delta = \frac{1}{v^-}\int (F_{n0} - F_{s0})\, dv^-. \tag{12}$$

We note that for $\theta_0 = 0$ the expression for $H_{cr}$ coincides with the one obtained by Ginzburg.[1]

The direction of the external field $\mathbf{H}_0$ will always be taken as positive so that we must have $H_{cr} \geq 0$. As far as the spontaneous magnetization $\mathbf{M}_0$ is concerned, two cases are possible.

(1) The case where $\cos\theta_0 \geq 0$. Then

$$H_{cr} = -4\pi M_0\cos\theta_0 + R. \tag{13}$$

In order that $H_{cr}$ be real and positive, the following inequality must be satisfied

$$8\pi\Delta \geqslant (4\pi M_0)^2[(1-n_1)\cos^2\theta_0 + (1-n_2)\sin^2\theta_0]. \tag{14}$$

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*Since at infinity the field $\mathbf{H} - \mathbf{H}_0$ is the field of a dipole with moment $\mu$, the same result for $\Phi_s$ can be obtained by using the equation (see Ref. 3).

$$\Phi_s = \int F_{s0}\, dv - \frac{v^-}{2}\mu\,\mathbf{H}_0.$$

To violate this inequality we must have $\Phi_n - \Phi_s < 0$, i.e., the superconducting phase must be thermodynamically unfavorable. If Eq. (14) is satisfied the superconducting state is possible* for $H_0 < H_{cr}$, and for $H > H_{cr}$ the normal state.

(2) The case where $\cos\theta_0 \leq 0$. Then there are, generally speaking, two critical fields possible,

$$H_{cr1} = 4M_0 |\cos\theta_0| - R, \quad H_{cr2} = 4\pi M_0 |\cos\theta_0| + R. \quad (15)$$

In order that such critical fields can exist it is necessary that the following conditions be fulfilled

$$(4\pi M_0)^2 (1 - n_2) \sin^2\theta_0 \leqslant 8\pi\Delta; \quad (16)$$

$$(4\pi M_0)^2 [(1 - n_1)\cos^2\theta_0 + (1 - n_2)\sin^2\theta_0] \geqslant 8\pi\Delta. \quad (17)$$

Condition (16) is necessary for the possibility of the existence of superconductivity. If it is violated, we get $\Phi_n - \Phi_s < 0$, i.e., the normal state will be more favorable. If (16) is satisfied, but condition (17) is violated, $H_{cr1}$ turns out to be negative, but $\Phi_n - \Phi_s > 0$ for $H_0 < H_{cr2}$. This means that the critical field $H_{cr1}$ does not exist and superconductivity is possible in a field $H_0 \leq H_{cr2}$. If both conditions (16) and (17) are fulfilled, the superconducting state is possible only in the range of fields $H_{cr1} \leq H_0 \leq H_{cr2}$. In the last case it is clearly necessary that the field $H_{cr1}$ directed against the spontaneous magnetization $M_0$ is less than the coercive force of the sample $H_c$, that is, we must have

$$\{8\pi\Delta(1 - n_1) - (4\pi M_0)^2 (1 - n_1)(1 - n_2)\sin^2\theta_0\}^{1/2} > $$
$$> 4\pi M_0 (1 - n_1) |\cos\theta_0| - H_c. \quad (18)$$

**5.** Let us now go over to a discussion of the conconditions which we have obtained for the existence of a superconducting state in ferromagnetic specimens. The quantity $8\pi\Delta \equiv (H^0_{crM})^2$ which occurs in Eqs. (14), (16), and (17) is clearly the square of the critical magnetic field, as it was in a non-ferromagnetic metal with the same difference $\Delta = F_{n0} - F_{s0}$, as also for the ferromagnetic considered. For the known superconducting elements the value of $H^0_{crM}$ at $T = 0$ varies from several tens to several thousands of oersteds. Below we shall assume $H^0_{crM} \sim 100$ oersted and $B_0 = 4\pi M_0 \sim 10^4$ gauss to obtain some estimates. The quantity $n_1$ varies from zero for a cylindrical specimen to unity for a specimen in the shape of a thin disk. The quantity $n_2$ varies from zero (disk) to $1/2$ (cylinder).

Let us consider the case $\cos\theta_0 > 0$ and condition (14). It is easily seen that for the values of $H^0_{crM}$ and $B_0$ chosen by us, it is necessary for the possible existence of a superconducting state to take a disk-shaped specimen with $1 - n_1 \lesssim 10^{-4}$, magnetized perpendicularly to the plane of the disk, while the angle $\theta_0$ of the deflection of $\mathbf{M}_0$ from the direction of $\mathbf{H}_0$ may not exceed $\theta_0 \sim 10^{-2}$. In the case of prolate specimens or for angles $\theta_0 > 10^{-2}$, the superconducting state is possible only for elements with an anomalously large value of $\Delta = F_{n0} - F_{s0}$ or else with an anomalously small value of $B_0 = 4\pi M_0$ which strongly decreases the probability of observing the superconductivity of ferromagnetics.

In the case $\cos\theta_0 < 0$ the necessary condition (16) also requires that the angles are small, $\theta_0 \lesssim 10^{-2}$. Condition (17), i.e., the condition for the existence of the critical field $H_{cr1}$, can be satisfied only for not too oblate specimens (or for anomalous values of $\Delta$ and $B_0$). In that case it is necessary to take also into consideration condition (18) which in several cases apparently can be realized.

In the case of prolate specimens the known values of $M_0$ and $H_c$ for ferromagnetic elements do not allow any hope for the observation of superconductivity in bulk specimens.

For very oblate specimens with $1 - n_1 \sim 10^{-4}$ superconductivity is possible in an arbitrarily small external field, limited by the requirement $H_0 < H_c$.

Thus we can summarize and say that the possibility to observe a superconducting state in bulk ferromagnetic samples is formally facilitated by using specimens with a large demagnetization factor.

In practice, however, it is impossible to obtain a single-domain sample with the quantity $1 - n_1 \sim 10^{-4}$, or in other words with the ratio of the transverse dimensions of the specimen to its thickness equal to $10^4$. In view of this it is necessary to analyze further the problem taking into account the role of domain structure, the energy of magnetic anisotropy, and so on.

In conclusion I want to use the opportunity to thank V. L. Ginzburg for valuable hints and for his interest in this paper.

*Indeed, as in the case of ordinary superconductors a pure superconducting state is possible only if the maximum field at the equator of the ellipsoid of revolution does not exceed the critical field, that is, in a field $H_0 < H_{crM}(1-n_1)$. In the range of fields $H_{crM}(1-n_1) < H_0 < H_{crM}$ the intermediate state is realized

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# *APPLICATION OF THE METHODS OF QUANTUM FIELD THEORY TO A SYSTEM OF BOSONS*

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It is shown that the techniques of quantum field theory can be applied to a system of many bosons. The Dyson equation for the one-particle Green's function is derived. Properties of the condensed phase in a system of interacting bosons are investigated.

## 1. INTRODUCTION

IN recent years Green's functions have been widely used[1] in quantum field theory, and in particular in quantum electrodynamics. This has made possible the development of methods[2] which escape from ordinary perturbation theory. The method of Green's functions has also been shown* to be applicable to many-body problems. In such problems the one-particle Green's function determines the essential characteristics of the system, the energy spectrum, the momentum distribution of particles in the ground state, etc.[3]

The present paper develops the method of Green's functions for a system consisting of a large number N of interacting bosons. The special feature of this system is the presence in the ground state of a large number of particles with momentum $\mathbf{p} = 0$ (condensed phase), which prevent the usual methods of quantum field theory from being applied. We find that for large N the usual technique of Feynman graphs can be used for the particles with $\mathbf{p} \neq 0$, while the condensed phase (we show that it does not disappear when interactions are introduced) can be considered as a kind of external field.

The Green's function is expressed in terms of three effective potentials $\Sigma_{ik}$, describing pair-production, pair-annihilation and scattering, and in terms of a chemical potential $\mu$. This is the analog of Dyson's equation in electrodynamics.[4,1] Some approximation must be made in the calculation of $\Sigma_{ik}$ and $\mu$. If these quantities are computed by perturbation theory, the quasi-particle spectrum of Bogoliubov[5] is obtained. In the following paper[6] we evaluate $\Sigma_{ik}$ and $\mu$ in the limit of low density.

*Private communication from A. B. Migdal.

## 2. STATEMENT OF THE PROBLEM. FEYNMAN GRAPHS

We consider a system of N spinless bosons with mass $m = 1$, enclosed in a volume V. We suppose N and V become infinite, the density $N/V = n$ remaining finite. A summation over discrete momenta is then replaced by an integral according to the rule

$$\sum_{\mathbf{p}} \rightarrow (2\pi)^{-3} V \int d\mathbf{p}.$$

The Hamiltonian of the system is $H = H_0 + H_1$, where

$$H_0 = \frac{1}{2}\int \nabla \Psi^+(\mathrm{x}) \nabla \Psi(\mathrm{x})\, d\mathrm{x} = \sum_{\mathbf{p}} \varepsilon^0_{\mathbf{p}} a^+_{\mathbf{p}} a_{\mathbf{p}}; \quad \varepsilon^0_{\mathbf{p}} = \frac{p^2}{2}, \tag{2.1}$$

$$H_1 = \frac{1}{2}\int \Psi^+(\mathrm{x}) \Psi^+(\mathrm{x}') U(\mathrm{x} - \mathrm{x}') \Psi(\mathrm{x}') \Psi(\mathrm{x})\, d\mathrm{x}\, d\mathrm{x}' = \frac{1}{2V} \sum_{\mathbf{pp'q}} U_{\mathbf{q}} a^+_{\mathbf{p}} a^+_{\mathbf{p'}} a_{\mathbf{p'-q}} a_{\mathbf{p+q}}. \tag{2.2}$$