THEORY OF THE MASER AND MASER FLUCTUATIONS

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The maser is considered as an oscillating system with one degree of freedom, the steady-state oscillations of which are descrbied by the Basov-Prokhorov equations. It is shown that there is a "soft" mode and an analog of the "hard" mode which is preceded by a region of noise generation. The natural line width due to thermal noise is determined and found to be 10^{-4} cps for an ammonia maser at room temperature.

1. The general form of the oscillating equation for a maser is represented by the system:

$$L_k(E + 4\pi\sigma) = 0;\ L_m(E) = \sigma,$$

where E is the field in the oscillator circuit and σ is the molecular polarization. The first equation is the usual differential circuit equation while the second applies to the system of molecules. The analysis of the second equation is extremely difficult and has not been carried out at the present time. However, certain problems in the theory of maser oscillations can be solved by means of the Basov-Prokhorov equation[1] which describes the steady-state oscillations and is a circuit equation with a complex dielectric constant ϵ associated with the beam of molecules

$$\frac{d^2V}{dt} + \frac{\omega_0}{Q}\frac{dV}{dt} + \frac{\omega_0^2}{\varepsilon}V = 0.$$

This equation can be obtained from the system written above if the well-known relation $E(\epsilon - 1)/4\pi = \sigma$ is applied in the second equation; in the present case this relation applies only in the case of sinusoidal oscillations because of saturation effects.

Fluctuations in amplitude and frequency in a maser are a result of thermal noise $\xi(t)$ with a spectral density

$$\omega_\xi = 4r\Theta;\ \Theta = \hbar\omega/2 + [e^{\hbar\omega/kT} - 1]^{-1}\hbar\omega$$

and the "shot" effect associated with the dielectric constant of the gas, i.e., fluctuations in ϵ. When these fluctuations are taken into account the equation is written in the form

$$\frac{d^2(V\varepsilon)}{dt^2} + \frac{\omega_0}{Q}\frac{d(V\varepsilon)}{dt} + \omega_0^2 V = \omega_0^2\xi(t).$$

It can be shown that the random dependence of ξ on t gives rise to a fluctuating emf (containing the derivatives $d\xi/dt$ and $d^2\xi/dt^2$) which is negligibly small compared with $\xi(t)$ even when T = 0. However, the parametric effect of the fluctuations in ϵ are important and are considered below. The final equation is

$$\frac{d^2V}{dt^2} + \omega_0^2 V = \mu\left[\omega_0 Q\left(1 - \frac{\varepsilon_1}{|\varepsilon|^2}\right)V - Q\left(\frac{\omega_0}{Q} + \frac{\omega_0^2}{\omega}\frac{\varepsilon_2}{|\varepsilon|^2}\right)\frac{dV}{dt}\right] + \omega_0^2\frac{\varepsilon_1}{|\varepsilon|^2}\xi(t), \tag{1}$$

where $V = A_0 \exp i\omega t$ is the steady-state sinusoidal voltage across the condenser, Q is the quality factor of the circuit, ω_0 is the natural resonant frequency of the circuit in the absence of molecules, $\epsilon = \epsilon_1(\omega, E_0, t) - i\epsilon_2(\omega, E_0, t)$ is the dielectric constant of the gas averaged over the volume, which is time dependent because of the shot effect, $E_0 = A/d$ is the field in the condenser, and $\mu = 1/Q$.

Equation (1) can be used only when V changes in a quasi-static way. Consequently the application of Eq. (1) for finding fluctuations in V is valid if $\xi(t)$ is characterized by a narrow spectrum $\Delta\omega_\xi \ll \Delta\omega_e$ (where $\Delta\omega_e$ is the width of the absorption line of the gas) centered about the frequency of the steady-state oscillations. As is shown by the theory,[2] this spectrum determines completely the spectral line width of the maser.

The solution of Eq. (1) is written in the form $V = A\cos\vartheta$ and the fluctuations are analyzed in accordance with the Bershtein method.[3] The usual transformations[3,4] yield the following equations:

$$\begin{aligned}
\frac{d\vartheta}{dt} &= \Psi(A,\dot\vartheta) - \frac{\omega_1}{A}\xi(t)\cos\omega_1 t;\\
\frac{dA}{dt} &= \Phi(A,\dot\vartheta) - \omega_1\xi(t)\sin\omega_1 t,\\
\Psi(A,\dot\vartheta) &= \omega_0 - \frac{\omega_0}{2}\left(1 - \frac{\varepsilon_1}{|\varepsilon|^2}\right);\\
\Phi(A,\dot\vartheta) &= -\frac{A}{2}\left[\frac{\omega_0}{Q} + \frac{\omega_0^2}{\omega}\frac{\varepsilon_2}{|\varepsilon|^2}\right],
\end{aligned} \tag{2}$$

where $\omega = \dot{\vartheta}$ and ω_1 are respectively the instantaneous and mean statistical frequencies of the steady-state oscillations.

2. Using the well-known expression[1] for the polarizability $\kappa_1(\omega, t)$ and $\kappa_2(\omega, t)$ and the distribution over time of flight $P(\tau) = (\tau/\tau_0^2) \times \exp(-\tau/\tau_0)$, the dielectric constant of the beam of gas molecules in the circuit, averaged over the volume, can be written

$$\varepsilon_2(\omega, E_0, t) = \frac{4\pi M(t)}{Sl} \int_0^\infty P(\tau) \frac{d\tau}{\tau} \int_0^\tau \varkappa_2(\omega, t)\, dt = -\frac{4\pi\hbar^3\beta\tau_0}{d^2 Sl} \frac{M(t)}{u^2+1+\Delta^2}, \quad \varepsilon_1(\omega, E_0, t) = 1 - u|\varepsilon_2|, \tag{3}$$

where $u = (\omega_2 - \omega)\tau_0$, ω_2 is the frequency of the photon, $\Delta^2 = E_0^2\beta\tau_0^2$, d is the dipole moment of the molecules, S and ℓ are the cross section and length of the beam of molecules and M(t) is the number of molecules in the condenser at any given instant of time.

The fluctuations in M(t) are determined by the shot effect in the dielectric constant. For a beam with the distribution denoted above by $P(\tau)$, the spectral density in the fluctuations of the number of particles $m(t) = M(t) - \overline{M}$ is

$$W_m(F) = 2\overline{M}\tau_0 (3 + \Omega^2\tau_0^2)/(1 + \Omega^2\tau_0^2), \quad \Omega = 2\pi F. \tag{4}$$

3. The mean values of the amplitude A_0 and the frequency $\vartheta_1 = \omega_1$ are found from the conditions $\Phi(A_0, \omega_1) = 0$ and $\omega_1 = \Psi(A_0, \omega_1)$. From the first of these, to an accuracy of μ^2, we find the equation for power as a function of tuning and oscillation frequency

$$\Delta^2 = \eta - 1 - u^2. \tag{5}$$

The second equation yields an equation for the circuit frequency (or oscillation frequency if the maser is excited)

$$u^3 - u^2 v + u(1 + \Delta^2 + R\eta) - v(1 + \Delta^2) = 0, \tag{6}$$

where $\eta = (8\pi\hbar^3/S\ell d^2)\tau_0^2\overline{N}Q$ is a dimensionless parameter which characterizes the excitation of the maser ($\overline{N}$ is the mean flux of molecules per unit time), $V = (\omega_2 - \omega_0)\tau_0$, $R = \omega_2\tau/2Q$ is the ratio of the quality factor of the absorption line of the gas $Q_\ell = \omega_2\tau_0/2$ to the figure of merit of the circuit; usually R is approximately $10^3 - 10^4$.

The mean amplitude and frequency are determined from the solution which satisfies Eqs. (5) and (6). From Eq. (5) we obtain the values at which self-excitation is achieved $|u| \leq \sqrt{\eta - 1}$. To determine the stability of these oscillations we write Eq. (2) with $\xi(t) = 0$ in the form

$$\omega_1 = \Psi_1(A_0), \quad dA_0/dt = \Phi_1(A_0, \Psi(A_0)). \tag{7}$$

In accordance with the general rule the oscillations are stable when

$$\frac{d\Phi_1}{dA_0} = p_0 = \left(p + \frac{p_1 q}{1 - q_1}\right) < 0, \tag{7}$$

where $p = \partial\Phi/\partial A$, $p_1 = \partial\Phi/\partial\omega$, $q = \partial\Psi/\partial A$, $q_1 = \partial\Psi/\partial\omega$ for $\omega = \omega_1$ and $A = A_0$. An examination of this equation indicates that when $\eta \leq 2$ the region of stable oscillations coincides with the excitation region $|u| \leq \sqrt{\eta - 1}$; when $\eta > 2$ the region of stable sinusoidal oscillations is smaller than the region of self-excitation and is equal to $|u| \leq \sqrt{\eta/2}$. In the unstable region $\sqrt{\eta/2} \leq |u| \leq \sqrt{\eta - 1}$ one expects noise oscillations with a spectrum of absorption lines or relaxation oscillations in which both frequency and amplitude change. When $\eta \leq 2$ there is a "soft" mode of production of harmonic oscillations depending on u or η while for $\eta \geq 2$ there is an analog to a "hard" model.

4. To determine the fluctuations in amplitude and phase we set $\vartheta = \omega = \omega_1 + \nu(t)$; $A = A_0 + A_0 \times \alpha(t)$, where $\nu(t)$ and $A_0\alpha(t)$ are the fluctuations in frequency and amplitude. Substituting these in Eq. (2) and taking $M(t) = \overline{M} + m(t)$, after making a series expansion and separation of variables, limiting ourselves to linear terms we have

$$\frac{d\nu}{dt} = p_0\nu + \frac{p_2 q - p q_2}{1 - q_1} m + \frac{q_2}{1 - q_1}\frac{dm}{dt} + \frac{1}{1 - q_1}\left[\frac{p}{A_0}\xi\omega_1\cos\omega_1 t - q\xi\omega_1\sin\omega_1 t - \frac{d}{dt}\left(\xi\frac{\omega_1}{A_0}\cos\omega_1 t\right)\right],$$

$$\frac{d\alpha}{dt} = p_0\alpha + \frac{p_m}{A_0} m - \frac{p_1}{1 - q_1}\xi\frac{\omega_1}{A_0}\cos\omega_2 t - \xi\frac{\omega_1}{A_0}\sin\omega_1 t, \quad p_2 = \partial\Phi/\partial M; \quad q_2 = \partial\Psi/\partial M; \quad p_m = p_2 + p_1 q_2/(1 - q_1). \tag{8}$$

It can be shown that the factor $p_2 q - p q_2$ is identically equal to zero, i.e., fluctuations in the number of particles do not produce fluctuations in the frequency of oscillation [assuming that Eq. (3) applies for ϵ]. This is easily explained by the effect of saturation more precisely by the dependence of the frequency of oscillation (by virtue of ϵ_1) not only on M but also on E_0. From Eq. (8) we obtain an expression for the spectral densities $\nu(t)$ and $\alpha(t)$ (for $u \approx 0$; $\eta > 1$)

$$W_\alpha(F) = \frac{\eta^2}{\overline{N}(\eta - 1)^2} + \frac{\Theta\eta^2}{P(\eta - 1)^2}; \quad W_\nu(F) = \frac{\Theta\omega_2^2}{R^2 P Q^2}, \tag{9}$$

where P is the oscillation power of the maser. The amplitude fluctuations are produced by the shot effect in the flux of molecules (first term) as well as thermal fluctuations (second term). The frequency fluctuations are produced only by thermal effects. In an ammonia maser with the circuit at room temperature $kT \gg \hbar\omega$ and $\Theta = kT$; setting $\tau_0 = 10^{-4}$ sec, $P = 10^{-9}$ watts, from Eq. (9) the spectral line width is found to be[5]

$$\Delta F = W_\nu(0)/4\pi = (kT/\hbar\omega_2)(1/\pi\bar{N})\tau_0^{-2} \simeq 10^{-4} \text{ cps}.$$

At low circuit temperatures, if millimeter waves are considered (or absolute zero for any wavelength) $\hbar\omega_1 \gg kT$ and $\Theta = \hbar\omega_2/2$, yielding $\Delta F = \omega_2^2/8\pi Q_1^2$ for the line width due to "zero field" fluctuations.

Equation (9) shows that the fluctuations in the frequency of a maser are $R^2 \sim 10^6 - 10^8$ times smaller than the fluctuations associated with the thermal noise in an equivalent vacuum tube oscillator. The monochromaticity and stability of maser oscillations are explained by the automatic frequency-regulating effect which results from the dependence of ϵ on the frequency of oscillation. In accordance with Eq. (3), an increase in frequency by an amount ν_0 leads to an increase in ϵ_1, thereby producing a compensating reduction of frequency by an amount ν_1; the resulting frequency displacement is

$$\nu = \nu_0 + \nu_1, \quad \nu_1 = \frac{d}{d\varepsilon_1}\left(\frac{\omega_0}{V\varepsilon_1}\right)\frac{d\varepsilon_1}{d\omega}\nu \simeq q_1\nu.$$

Thus $\nu = \nu_0/(1 - q_1) \approx \nu_0/R$ (it can be shown that $1 - q_1 \cong q_2 \approx R$). If ν_0 is the fluctuation in frequency which takes place without self-compensation (fluctuations in an ordinary oscillator), Eq. (9) is easily obtained from the expression $\overline{\nu^2} = \overline{\nu_0^2}/R^2$.

If the frequency of the fluctuations is such that $2\pi F > \tau_0^{-1}$ the self-regulating effect of ϵ_1 does not operate (ϵ_1 cannot "follow" variations in ω and A) and one expects that at these frequencies W(F) will be R^2 times larger than indicated by Eq. (9). However, this situation does not lead to a broadening of the line but merely increases the intensity in the wings. The spectral line width has been considered in Ref. 6 by another method but the results obtained there differ somewhat from those described here.

A detailed description of the present work is given in Ref. 7.

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