ductivity $\sigma = \sigma(\omega)$. Using the connection between these quantities and the optical constants, it is then easy to find the dispersion formulas for the index of refraction $n = n(\omega)$ and the extinction coefficient $k = k(\omega)$. If in these calculations we keep the third order terms, we have a derivation of the dispersion formulas with approximate inclusion of the effects of interaction between the electrons.

This opens up possibilities for the development of a many-electron theory of the optical properties of crystals within the framework of the quasi-homopolar approximation.

The practical realization of this program encounters mathematical difficulties in connection with the diagonalization of the operator (5.1).

Using Tiablikov's method of approximate diagonalization,[2,3] which is correct for weak perturbations of the system, it is possible in principle to carry out the indicated program for some particular case, for example for a ferromagnetic crystal in a state close to saturation.

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CONTRIBUTION TO THE THEORY OF ANTIFERROMAGNETISM AT LOW TEMPERATURES

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The energy spectrum of a uniaxial antiferromagnetic substance is obtained without the assumption of nominal magnetization of the spin sublattices in the ground state.

THE existing theory of antiferromagnetism[1] is based on the hypothesis (first proposed by Landau[2]) that an antiferromagnetic substance can be described in terms of two or more magnetic moments, which compensate each other in the absence of magnetic field. The construction of the microscopic theory of antiferromagnetism ordinarily begins with the Heisenberg model of exchange interaction and the assumption of a regular ordering of "left" and "right" spins in the ground state.[3,4]

As has been remarked by Landau, such an assumption is in contradiction with quantum mechanics: the spin component of an individual atom in a prescribed direction cannot have a definite value, because of the exchange interaction. On the other hand it appears that the experimental data do not contradict the results obtained by the use of this model.

The purpose of the present paper is to show that the energy spectrum of an antiferromagnetic substance, and thus all of its thermodynamic functions, can be obtained in a phenomenological way just from the assumption of two (or more) sublattices, without postulating nominal magnetization of each of the sublattices in the ground state.

1. The assumption of the existence of two (or several) sublattices can be reduced to the assumption that the state of an antiferromagnetic substance is characterized by the specification of two (or several) magnetic moments $\mathbf{M}_i(\mathbf{r})$ at each point. Here one picks out as the ground state the state with homogeneous values of each of the magnetic

moments, $\mathbf{M}_{i0}$, such that $\sum \mathbf{M}_{i0} = 0$. Such a description is clearly legitimate at low temperatures, for which only long-wave vibrations of the spin system are excited and the atoms of an elementary cell can be regarded as situated at a single point.

In what follows we shall consider for simplicity an antiferromagnetic substance consisting of two sublattices with magnetic moments $\mathbf{M}_1(\mathbf{r})$ and $\mathbf{M}_2(\mathbf{r})$. This treatment is surely applicable to uniaxial antiferromagnetic substances.

We write the Hamiltonian of the system in the following form:

$$\mathscr{H} = \int dV \left\{ \alpha \mathbf{M}_1 \mathbf{M}_2 - \mathbf{M}_1 \mathbf{H} - \mathbf{M}_2 \mathbf{H} \right. \\ \left. + \frac{1}{2} \lambda (M_{1x}^2 + M_{1y}^2 + M_{2x}^2 + M_{2y}^2) \right. \tag{1}$$

where α, β, β_{12} are constants connected with the exchange interaction, and the last three terms describe the exchange interaction caused by the inhomogeneity of the spin system. As will be seen below, in antiferromagnetic substances $\beta - \beta_{12} > 0$. Furthermore it can be shown that

$$\beta,\ \beta_{12} \sim \alpha a^2 \sim \Theta_C / M^2 a,$$

where Θ_C is the Curie temperature and a is the lattice constant; if $M \sim \mu/a^3$ (μ is of the order of magnitude of the Bohr magnetron), then β, $\beta_{12} \sim a^2 \Theta_C/\mu M$ and $\alpha \sim \Theta_C/\mu M \gg 1$. In the general case the term involving the spatial inhomogeneity of the magnetic moments is of the form

$$\beta_{iklm}^{ss'} (\partial M_{si} / \partial x_k)(\partial M_{s'l} / \partial x_m).$$

For a uniaxial substance, in virtue of the isotropic nature of the exchange interaction, we get the expression that appears in the Hamiltonian (1).

In the above, λ is the constant of the magnetic anisotropy (the z axis is taken along the preferred direction). We note that $\lambda \ll \alpha$, since the magnetic anisotropy is due to relativistic interactions: $\lambda \sim \alpha (v\ \ c)^2$, where v is the speed of the electrons and c is that of light. We have omitted from the Hamiltonian terms proportional to the squares of the magnetic moments, M_1^2 and M_2^2, since they do not make any contribution to the equations of motion of the magnetic moments (see below).

The magnetic field H is composed of the constant homogeneous external field $\mathbf{H}_0$ and the magnetic field h of the spin waves, which satisfies the equations of magnetostatics,

$$\operatorname{curl} \mathbf{h} = 0, \quad \operatorname{div} \mathbf{h} = -4\pi \operatorname{div} (\mathbf{M}_1 + \mathbf{M}_2). \tag{2}$$

Starting with the Hamiltonian (1), we write the equations of motion of the magnetic moments:

$$\partial \mathbf{M}_s / \partial t = g [\mathbf{M}_s \times \mathbf{H}_{\text{eff}}^s]; \quad \mathbf{H}_{\text{eff}}^s = -\delta \mathscr{H} / \delta \mathbf{M}_s, \quad (s = 1,\ 2), \tag{3}$$

where $\delta\mathscr{H}/\delta \mathbf{M}_s$ is the variational derivative of the Hamiltonian with respect to the magnetic moment and g is the gyromagnetic ratio ($g\hbar = \mu$).

Using Eqs. (2) and (3) (in analogy with the procedure used by Herring and Kittel[5]), we can find the magnetic branch of the energy spectrum of the antiferromagnetic substance in the neighborhood of the ground state. To do this we linearize (2) and (3), substituting for $\mathbf{M}_1$ and $\mathbf{M}_2$ the expressions

$$\mathbf{M}_1 = \mathbf{M} + \boldsymbol{\mu}_1; \qquad \mathbf{M}_2 = -\mathbf{M} + \boldsymbol{\mu}_2. \tag{4}$$

Taking $\mathbf{H}_0$ along the z axis and assuming that all the variable quantities ($\boldsymbol{\mu}$ and $\mathbf{h}$) are proportional to $e^{-i(\omega t - \mathbf{kr})}$, we get:

$$\begin{gathered} i\omega (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) - g [\mathbf{H}_0 \times (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)] \\ = g \{\lambda + (\beta - \beta_{12}) k^2\} [\mathbf{M} \times (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)], \\ - i\omega (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) + g [\mathbf{H}_0 \times (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)] \\ = 2g \mathbf{M} \times \{(\alpha - \lambda - (\beta - \beta_{12}) k^2)(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) \\ - (8\pi \mathbf{k} / k^2) [\mathbf{k} \times (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)]\} \end{gathered} \tag{5}$$

Since the constant α is very large, the second equation of (5) can be considerably simplified:

$$i\omega (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) - g [\mathbf{H}_0 \times (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)] = -2g\alpha [\mathbf{M} \times (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)]. \tag{6}$$

In fact, $\lambda \ll \alpha$ and

$$\beta k^2 \sim (\Theta_C / \mu M)(ak)^2 \sim \alpha (ak)^2 \ll \alpha,$$

since we are of course interested only in the long-wavelength vibrations ($ak \ll 1$).

Neglect of the last term in the second equation of (5) is equivalent to the neglect of the magnetic field of the spin wave. This means that in an antiferromagnetic substance the proper magnetic field of the spin waves never leads to a change of the dispersion law, as occurs in a ferromagnetic substance (cf. Refs. 5 and 6).

From Eqs. (5) and (6) we have

$$\omega_{1,2} = \sqrt{2g^2 \alpha M^2 [\lambda + (\beta - \beta_{12}) k^2]} \pm g H_0; \tag{7}$$

and the energy $\epsilon = \hbar\omega$ of the spin wave is given by

$$\varepsilon_{12} = \mu M \sqrt{2\alpha [\lambda + (\beta - \beta_{12}) k^2]} \pm \mu H_0. \tag{8}$$

From (7) or (8) it can be seen that the difference $\beta - \beta_{12}$ must be positive, since in the opposite case the ground state would have a finite value of k, i.e., the ground state would be inhomogeneous. This fact enables us to introduce the following notation:*

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*Essentially this is a definition of the two quantities H_a and θ_C. The former is the magnetic field that would have to be applied along the axis of easiest magnetization to make the antiferromagnetic state thermodynamically unstable;[3] the latter is of the order of the Curie temperature.

$$\sqrt{2\alpha\lambda}\,M = H_a, \quad \sqrt{2\alpha(\beta - \beta_{12})}\,\mu M / a = \Theta_C. \tag{9}$$

We then have from Eq. (8)

$$\varepsilon_{1,2} = \sqrt{(\mu H_a)^2 + \Theta_C^2 (ak)^2} \pm \mu H_0 \quad (H_0 < H_a). \tag{10}$$

For $H_0 = 0$ there is a two-fold degeneracy:

$$\varepsilon_{1,2} = \sqrt{(\mu H_a)^2 + \Theta_C^2 (ak)^2}. \tag{11}$$

For $k = 0$ the energy $\epsilon_{1,2}$ of the spin wave is equal to μH_a, and for sufficiently large k it is proportional to k:

$$\varepsilon_{1,2} \approx \begin{cases} \mu H_a + {}^1\!/_2\, \Theta_C (\Theta_C / \mu H_a)(ak)^2, & ak \ll \mu H_a / \Theta_C, \\ \Theta_C (ak), & ak \gg \mu H_a / \Theta_C. \end{cases} \tag{12}$$

Using Eq. (11) for the energy of a spin wave, we find by the usual formulas of statistical physics the spin part of the specific heat of an antiferromagnetic substance:

$$C_s = \begin{cases} (1/\pi\sqrt{2\pi})\, R\, (\mu H_a / \Theta_C)^3 (\mu H_a / T)^{1/2} e^{-\mu H_a / T} & (T \ll \mu H_a), \\ (4\pi^2/15)\, R\, (T/\Theta_C)^3 & (\Theta_C \gg T \gg \mu H_a). \end{cases} \tag{13}$$

2. For the determination of the temperature dependence of the magnetic susceptibility ($\chi_{\parallel}$ and $\chi_{\perp}$) we must know in particular the energy spectrum of the antiferromagnetic substance in a magnetic field perpendicular to the axis of easiest magnetization.

If we take the x axis along the magnetic field, in the ground state the magnetic moments of the two sublattices have the following components ($\lambda \ll \alpha$):

$$M_{1x} = M_{2x} = H / 2\alpha; \quad M_{1y} = M_{2y} = 0; \quad M_{1z} = -M_{2z} = \sqrt{M^2 - M_{1x}^2}. \tag{14}$$

We note that here each of the moments is parallel to its own effective field. Moreover, the expressions (14) make clear the physical meaning of the constant α: $1\ \alpha = \chi_{\perp 0}$, where $\chi_{\perp 0}$ is the value of the perpendicular component of the magnetic susceptibility at temperature zero.

Using linearized equations of motion analogous to the preceding, we find the energy spectrum in this case (H perpendicular to the axis of easiest magnetization):

$$\omega_1 = gM\sqrt{2\alpha(\lambda + (\beta - \beta_{12})k^2)}, \quad \omega_2 = gM\sqrt{2\alpha[\lambda + (\beta - \beta_{12})k^2] + (H/M)^2} \tag{15}$$

or (in the previous notation)

$$\varepsilon_1 = \sqrt{(\mu H_a)^2 + \Theta_C^2 (ak)^2}, \quad \varepsilon_2 = \sqrt{(\mu H_a)^2 + (\mu H)^2 + \Theta_C^2 (ak)^2}. \tag{16}$$

Knowledge of the dependence of the energy of the spin waves on the magnetic field [Eqs. (10) and (16)] enables us to find the dependence of the magnetic susceptibility on the temperature:

$$\chi_{ik} = -\partial^2\Omega / \partial H_i \partial H_k \big|_{H=0},$$

where $\Omega = T \sum_k \ln(1 - e^{-\epsilon_k/T})$ is the thermodynamic potential of the "spin wave gas."

After some simple calculations we get

$$\chi_{\parallel} = \begin{cases} \dfrac{1}{\pi\sqrt{2\pi}} \dfrac{\mu^2}{a^3\Theta_C} \left(\dfrac{\mu H_a}{\Theta_C}\right)^{3/2} \left(\dfrac{T}{\Theta_C}\right)^{1/2} e^{-\mu H_a/T} & (T \ll \mu H_a), \\ \dfrac{\mu^2}{3a^3\Theta_C} \left(\dfrac{T}{\Theta_C}\right)^2 & (\Theta_C \gg T \gg \mu H_a); \end{cases} \tag{17}$$

$$\chi_{\perp} = \begin{cases} \dfrac{1}{\alpha} + \dfrac{1}{(2\pi)^{3/2}} \dfrac{\mu^2}{a^3\Theta_C} \left(\dfrac{\mu H_a}{\Theta_C}\right)^{1/2} \left(\dfrac{T}{\Theta_C}\right)^{3/2} e^{-\mu H_a/T} & (T \ll \mu H_a), \\ \dfrac{1}{\alpha} + \dfrac{\mu^2}{12a^3\Theta_C} \left(\dfrac{T}{\Theta_C}\right)^2 & (\Theta_C \gg T \gg \mu H_a). \end{cases} \tag{18}$$

The temperature dependences found here for the specific heat and the magnetic susceptibility naturally agree with the results previously obtained by Néel by microscopic considerations.[1] Here they bring us to our goal of relating the phenomenological constants Θ_C and H_a, which we have introduced to the measurable quantities c_s, $\chi_{\parallel}$, and $\chi_{\perp}$.

3. One can, of course, deal in a similar way with more complicated magnetic structures, in particular ferrites, in which the magnetic moments of the sublattices do not compensate each other.

If there are no special reasons for supposing that the difference between the magnetic moments of the sublattices is small (as, for example, in the case of Fe_2O_3, according to Refs. 7 and 8), then on carrying out calculations similar to those above we easily find that the magnetic part of the spectrum consists of two branches (if there are two sublattices). One of these has a large activation energy, of the order of Θ_C and is of course not excited at low temperatures (this is the analog of the optical branch in the vibrations of compound lattices). The other one is analogous to the ordinary Bloch spin waves. For the case of two moments and at not too low temperatures it has the form (supposing $M_1 > M_2$):

$$\omega = \frac{2gM_1M_2}{M_1 - M_2}(\beta - \beta_{12})k^2. \tag{19}$$

Therefore the magnetic part of the specific heat must be proportional to $T^{3/2}$, as it is for ordinary ferromagnetic substances.

In conclusion the writers take this occasion to thank L. D. Landau and I. M. Lifshitz for valuable discussions on the questions considered here.

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SCATTERING OF DIRAC PARTICLES BY A SHORT-RANGE CENTER OF FORCE WITH DAMPING TAKEN INTO ACCOUNT

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The theory of radiation damping is used to investigate elastic scattering of Dirac particles by a stationary short-range center of force. An equation is obtained for the scattering cross section A relation [Eq. (5)] is established between the scattering phase shifts predicted by the theory of radiation damping for Dirac particles and relativistic spinless particles.

In the present work the theory of radiation damping is used to study elastic scattering of Dirac particles by an arbitrary short-range center of force. This has already been done[1] only for spinless particles (hereinafter we shall use the notation developed in that article and denote it by SK). In the present case we shall divide the wave functions into two groups according to the z component of the spin ($m_s = \pm\frac{1}{2}$), rather than according to the component of the spin in the direction of motion, as was done in SK. Then the fundamental integral equation [see Eq. (21) of SK] of the theory of radiation damping for elastic scattering of spin-$\frac{1}{2}$ particles becomes

$$(\varepsilon_n(l) - 1)H^{(n)}_{\mathbf{k}'\mathbf{k}}(l, m_{s'}, m_s) = \frac{kK}{8\pi^2 c\hbar i}\sum_{l', n', m_{s''}} \varepsilon_{n'}(l') \oint d\Omega'' H^{(n)}_{\mathbf{k}'\mathbf{k}''}(l, m_{s'}, m_{s''}) H^{(n')}_{\mathbf{k}''\mathbf{k}}(l', m_{s''}, m_s), \tag{1}$$