Since in this approximation

$$\mathrm{Im}\,\langle jlk\,|\,T'\,|\,jLp_1\Lambda q_1 s''\rangle = \rho\,\tfrac{1}{2}\,pM\,|\,\langle jl's'p\,|\,T'\,|\,jLp_1\Lambda q_1 s''\rangle\,|\,,$$

the right member contains experimentally measurable quantities.

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THE FEYNMAN PATH INTEGRAL FOR THE DIRAC EQUATION

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It is shown that, subject to certain assumptions about the motion of the electron, the Feynman path integral is identical with the propagation function of the Dirac equation.

1. INTRODUCTION

The problem considered below was initiated by Feynman and has been discussed in Refs. 1—4. The main result of the first paper[1] is the relation

$$R(x_1t_1\,|\,x_2t_2) = \int e^{iS\{x(t)\}/\hbar}\delta x(t). \tag{1}$$

Here $R(x_1t_1|x_2t_2)$ is the solution of the Schrödinger equation which becomes $\delta(x_1 - x_2)$ for $t_1 = t_2$;

$$S = \int_{t_1}^{t_2} L\{\dot{x}(t), x(t)\}\,dt$$

is the increase of the classical action along a path; and $\int \delta x(t)$ denotes integration (summation) over all trajectories $x(t)$ for which $x(t_1) = x_1$ and $x(t_2) = x_2$.

Together with certain other rules, the relation (1) gives a closed formulation of quantum mechanics. This formulation is mathematically equivalent to the usual formalism, but the Feynman approach has a number of advantages. In particular, the formal solution of any problem is obtained in the form of an infinitely multiple integral over the paths. There is a hope that just this approach will give a simple and perspicuous system of concepts and notations which will make it possible to get beyond the framework of perturbation theory in quantum mesodynamics.

Another virtue of the method is the space-time description and the absence of operators. One can follow in thought the motion of the particle—the motion turns out to be a special case of a Markov random process.

One further important advantage is the connection of the method with the Hamilton principle of least action. The formula (1) gives so to speak the probability of those virtual trajectories which are considered in the formulation of the principle of least action. And whereas the classical equation of motion is obtained firstly from a Lagrangian (specific information about the system) and secondly from the principle of least action, the quantum equation of motion in the nonrelativistic domain (the Schrödinger equation) is obtained firstly from the same classical Lagrangian and secondly from the Feynman principle, Eq. (1). The extension of this scheme to the relativistic domain is the purpose of the present paper. In analogy with the use of Hamilton's principle in classical physics we can expect that the principle (1) will be important for the solution of various theoretical problems.

But there is another side of the question: Strictly speaking, Eq. (1) is not applicable to any physical system except the electromagnetic field and the nonrelativeistic case, which is of little interest. Indeed, if we substitute in $S\{x(t)\}$ in Eq. (1) the classical relativistic Lagrangian, we do not get on the left-hand side the solution of the Dirac equation, above all because of the absence of the spinor indices in the classical picture.

But it turns out to be possible, even in the classical picture, to give a geometrical meaning to the spinor indices (see below) which makes it possible to obtain the Dirac equation from the Feynman principle. The considerations on which the proposed solution is based are explained in the conclusion. Use is also made of some ideas from Refs. 3—6.

2. A TWO-DIMENSIONAL ANALOGUE OF THE DIRAC EQUATION

In considering the problem of the relativistic formulation of the Feynman principle we use as an example the equation

$$[(\partial/\partial t + iA_t) + \sigma_1(\partial/\partial r - iA_r) + i\mu\sigma_3]\psi = 0, \quad \psi = \begin{matrix}\psi_1(r,t)\\ \psi_{-1}(r,t)\end{matrix}, \quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{2}$$

In this case the classical action has the form

$$\int(-\mu\sqrt{dt^2 - dr^2} + A_r\,dr - A_t\,dt). \tag{3}$$

This is, as it were, half of the Dirac equation, and the results of the analysis will remain in force for the electron. All of the developments for Eq. (2) are simple, and the paths can be drawn on paper.

For what follows it is convenient to take instead of Eq. (2) the more general equation

$$[(\partial/\partial t + iA_t) + \sigma_1(\partial/\partial r - iA_r) + \sigma_3(\partial/\partial s + i\mu)]\psi(r,t,s) = 0. \tag{4}$$

If the coefficients and the initial condition do not depend on s, then Eq. (4) reduces to Eq. (2).

We give the name of the propagation function $R(r, t, s\,|\,r', t', s')$ to the solution of the equation which reduces to the δ function at the initial time. This function has the property

$$\psi(r,t,s) = \int R(r,t,s\,|\,r',t',s')\psi(r',t',s')\,dr'\,ds'. \tag{5}$$

In order to construct the integral (1) for Eq. (4) we must first relate the propagation function of Eq. (4) to the classical action for small time intervals. Noting the results of Refs. 3—6, we shall seek this relation in a mixed representation, i.e., we shall number the columns of the propagation matrix with eigenvalues of the matrix σ_3 (which we denote by n, with $n = \pm 1$) and the rows with eigenvalues of the matrix σ_1 (which we denote by m, with $m = \pm 1$).

To avoid difficulties with continuous paths we shall assume that the particle can exist only at the nodal points of a space-time lattice with the period ϵ; then Eq. (4) is replaced by the corresponding equation in finite differences. The relationship expressing the Feynman principle for this equation has the form

$$R_{nm}(r+\varepsilon_r, t+\varepsilon, s+\varepsilon_s\,|\,r,t,s) = \eta_{nm}\exp\{-i\mu\varepsilon_s + iA_r\varepsilon - iA_t\varepsilon\}. \tag{6}$$

Here R_{nm} is the propagation function, $\epsilon_r = m\epsilon$ is the change of the coordinate, $\epsilon_s = n\epsilon$ is the change of the proper time, and

$$\eta_{nm} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}_{nm} = \pm\frac{1}{\sqrt{2}}.$$

Substituting Eq. (6) into Eq. (5) we get, for $\epsilon \to 0$, Eq. (4).

The paths of the particle are obtained from Eq. (6) in the following way. Suppose that at the time $t = 0$ the particle was at the point $r = 0$, $s = 0$ with the spinor index m, and that at the next instant of time, $t = \epsilon$, the spinor index is equal to n. From Eqs. (5) and (6) we get that at the instant $t = \epsilon$ the wave function is different from zero only at the point $r = m\epsilon$, $s = n\epsilon$. To the four possible combinations of initial and final indices there correspond the four paths shown in Fig. 1.

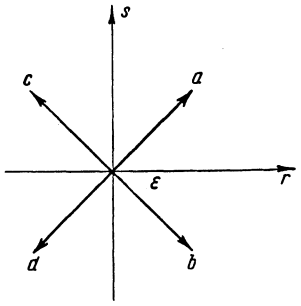

FIG. 1. (a) $n = 1$, $m = 1$; (b) $n = -1, m = 1$; (c) $n = 1$, $m = -1$; (d) $n = -1$, $m = -1$

Let us follow the path of the particle (Fig. 2). If $n = 1$ at the instant $t = 0$, the transitions a and c are possible, and at the instant $t = \epsilon$ the particle will be either at A or at B. If the particle has come to the point A, the transition has been of the type c and the particle has the spinor index $m = -1$. For $m = -1$ the possible transitions are c and d (Fig. 1), and consequently at the instant $t = 2\epsilon$ the particle will be either at C or at D. If the particle has come to the point D, then $n = -1$ and the possible transitions are b and d, and so on.

To each path in the space r, s there corresponds a definite succession of indices (and vice versa), and thus the geometrical significance of the spinor indices becomes clear. Namely, the two values of the number n denote two modes of intersection of the world line with the r, t plane, and $|\psi_1(r, t)|^2$ and $|\psi_{-1}(r, t)|^2$ are the probabilities for finding the particle moving through the point (r, t) in the directions of larger and smaller s.

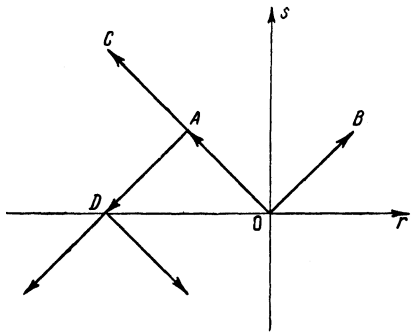

FIG. 2

This interpretation of necessity leads to the appearance of spinor indices on the propagation function. The transition $r', t' \to r, t$ occurs along one of the paths, and the prescription of the path means the prescription of the displacements along the path. Therefore the transition amplitude is obtained by addition of the amplitudes for the separate paths, and the amplitude for each path is obtained by multiplying together the amplitudes (6) of the displacements. If spinor indices are prescribed at the instants t' and t, for example n' and n, the transition is possible only along a particular type of path, Fig. 3, and only these paths contribute to the transition amplitude. The four amplitudes that form the matrix propagation function correspond to the four possibilities for the paths (see Fig. 3).

These considerations, i.e., the repeated application of Eq. (6), give for $t - t' = N\epsilon$

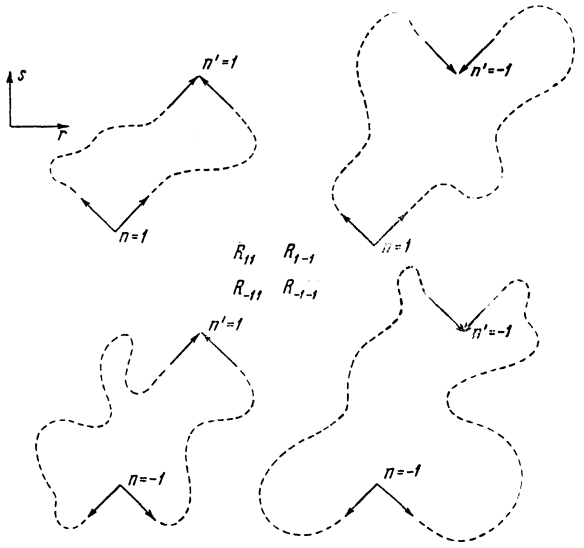

FIG. 3

$$R_{nn'}(r, t, s \mid r', t', s') = \sum_{\epsilon_r^k, \epsilon_s^k} E_{nn'} e^{iS}, \tag{7}$$

with the condition $\sum \epsilon_r^k = r - r'$, $\sum \epsilon_s^k = s - s'$; $\epsilon_r^k = m_k\epsilon = \pm\epsilon$ is the change of the coordinate of the particle during the time interval (t_{k+1}, t_k), with $t_k = t' + k\epsilon$, and $\epsilon_s^k = n_k\epsilon = \pm\epsilon$ is the change of the proper time during this same interval. The summation is taken over all the successive displacements, i.e., over all the paths in the space, and

$$E_{nn'} = \eta_{m_1}\eta_{m_1n_2}\eta_{n_2m_2}\eta_{m_2n_3}\ldots\eta_{n_Nm_N}\eta_{m_Nn'} = \pm 2^{-N}.$$

Unlike the case of Eq. (1), part of the paths occur with the opposite sign. But all the paths for which the sign of ϵ_s, i.e., the sign of the change of the proper time, does not change along the path, occur with the sign +. Only these paths remain in the nonrelativistic approximation

$$S = -\sum \mu \varepsilon_s^k + \sum A_r(t_k, r_k)\varepsilon_r^k - \sum A_t(t_k, r_k)\varepsilon = -\sum \mu (s_k - s_{k-1}) + \sum A_r(t_k, r_k)(r_k - r_{k-1}) - \sum A_t(t_k, r_k)(t - t_{k-1}),$$

$$S \to -\int \mu ds + \int A_r dr - \int A_t dt \quad \text{for } \varepsilon \to 0.$$

The propagation function $R(r, t|r', t')$ is the sum over paths with arbitrary change of the proper time. Therefore

$$R_{nn'}(r, t|r', t') = \int_{-\infty}^{\infty} R_{nn'}(r, t, s|r', t', s')\, d(s - s') = \sum_{\varepsilon_r^k \varepsilon_s^k} E_{nn'} e^{iS}, \quad \sum \varepsilon_r^k = r - r'. \tag{8}$$

The meaning of the factor $E_{nn'}$ is partly revealed if we go over to a new representation $\psi'_n = e^{i\pi n/4}\psi_n$ and introduce the notation $\delta n_k = n_{k+1} - n_k$; we get

$$R'_{nn'} = e^{i\pi(n-n')/4} R_{nn'} = \sum \exp\left\{iS + \frac{i\pi}{4}\sum m_k \delta n_k\right\}, \quad n_1 = n, \; n_{N+1} = n'. \tag{9}$$

From this it can be seen that the sign factor $E_{nn'}$ is the exponential of the sum along the path. In the classical action there appears an additional term depending on the signs of the displacement. This additional term can be shown to be equal to the quantity $(\pi h/2)(M - L)$, where M is the number of turns to the right and L the number to the left along the path; that is:

$$R'_{nn'} = \sum \exp\left\{iS + \frac{i\pi}{2}(M - L)\right\}. \tag{9'}$$

Here one needs to take account only of turns with change of n or only those with change of m ($\sum m_k \delta n_k = -\sum n_k \delta m_k$). In the latter case reflection in the line S = const of a part of the path bounded by that line does not change the action along the path. But on reflection left and right turns are interchanged, so that if the reflected part of the path contains a turn the new path occurs in the sum with the opposite sign, and the contributions of the two paths cancel. There remain only transitions that are asymmetrical with respect to the turns, and the particle turns out to be "rotating."

Here there is an analogy with the expression for the nonrelativistic propagation function for several electrons, $R = \sum \exp(iS + i\pi P)$, where P is the number of interchanges of the final coordinates. In the relativistic theory the antisymmetry with respect to interchanges is supplemented by an antisymmetry with respect to replacement of one of the left turns by a right turn.

Let us consider further the calculation of the sums (7) and (8). If there is no external field, i.e., $A_r = A_t = 0$, these sums are easily calculated by the method of Markov (the motion of the particle is a Markov process). In Eq. (8) we must replace the condition $\sum \epsilon_r^k = r - r'$ by the factor

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left\{i\left(\sum \varepsilon_r^k - r - r'\right)p'/\varepsilon\right\} dp'.$$

The expression under the sign of summation is the product of N two-rowed matrices. Let the transformation Q bring each of the matrices to diagonal form with eigenvalues λ_1 and λ_2. Then

$$R = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} Q^{-1} \exp\{-i(r - r')p'/\varepsilon\}\, dp', \quad \lambda_{1,2} = 1 \pm i\varepsilon\sqrt{\mu^2 + p^2}, \quad p = p'/\varepsilon.$$

For $\epsilon \to 0$, $N\epsilon = t$

$$R \to \frac{1}{2\pi}\int_{-\infty}^{\infty} Q \begin{pmatrix} \exp\{it\sqrt{\mu^2 + p^2}\} & 0 \\ 0 & \exp\{-it\sqrt{\mu^2 + p^2}\} \end{pmatrix} Q^{-1} \exp\{-i(r - r')p\}\, dp.$$

3. TRANSITION TO CLASSICAL MECHANICS

The transition to small h is made as in Ref. 1. In this connection, there appear in the relativistic domain new features that are of interest from the point of view of the relation between the classical and quantum pictures of the motion.

Let us go back to Eq. (7) and divide the time $t - t'$ into q intervals of size a, such that the external field can be regarded as constant for regions of space or time that are smaller than a (the speed of light is equal to 1). Let $r', r_1, \ldots, r_q = r$ and $s', s_1, \ldots, s_q = s$ be the coordinates and proper times of the particle at the instants $t', t' + a, \ldots, t' + qa = t$. One can divide the summation over the paths into two stages and first take the sum over all paths that pass through $r', r_1, \ldots, r$ and $s', s_1, \ldots, s$ and then the sum over all values of r_1 and s_1. Carrying out the first summation and setting $\epsilon = 0$, we get

$$R(r, t, s \mid r', t', s') = \int \exp\left\{i \sum (-\mu \Delta s_k + A_r(t_k, r_k)\Delta r_k - A_t(t_k, r_k)\Delta t_k)\right\} \prod_k \rho(\Delta s_k, \Delta r_k, \Delta t_k)\, dr_k ds_k,$$

$$\Delta s_k = s_k - s_{k-1}, \quad \Delta r_k = r_k - r_{k-1}, \quad \Delta t_k = t_k - t_{k-1}, \tag{10}$$

ρ will obviously be the propagation function of a free particle with $\mu = 0$.

For $h \to 0$ (i.e., $\mu \to \infty$), because of the rapid oscillation of the factor $\exp\{-i\mu(sk - s_{k-1})\}$, the contributions from regions with smooth variation of ρ cancel each other and there remain only paths along which ρ has a singularity, i.e., paths with $\Delta s_k^2 = \Delta t_k^2 - \Delta r_k^2$. In this approximation

$$R(r, t \mid r', t') = \int \exp\left\{i \sum \left(\pm \mu \sqrt{\Delta t_k^2 - \Delta r_k^2} + A_r \Delta r_k - A_t \Delta t_k\right)\right\} \prod_k \rho'(\Delta r_k, \Delta t_k)\, dr_k. \tag{11}$$

Here the largest contribution will be from the region with slow variation of the phase of the exponential function (we denote this phase by S), i.e., with $\partial S/\partial r_k = 0$. If the sign in front of the square root changes on the passage of the particle through r_k, then, as can be shown, there will be no stationary solution (if the field varies smoothly). There remain only paths for which the sign of the root does not change and $\partial S/\partial r_k = 0$, i.e., the principle of least action is satisfied.

Thus for $h \to 0$ the picture of the motion undergoes an essential change. Thus for the quantization of the relativistic equations, i.e., formally, for the reconstruction of the theory from its limiting case, the nonrelativistic Feynman principle or, what is the same thing, the replacement of the momenta by operators, turns out to be insufficient.

4. DERIVATION OF THE NONRELATIVISTIC FEYNMAN PRINCIPLE

The nonrelativistic approximation means that all distances are large in comparison with the Compton wavelength h/mc. Therefore, as in Sec. 3, $\mu \Delta s_k \to \infty$ and Eq. (10) goes over into Eq. (11). In the nonrelativistic approximation all the elements of the matrix ρ' in Eq. (11) go to zero except ρ_{11} if the plus sign is taken in front of the root, or ρ_{-1-1} if the minus sign is taken. Therefore, as in Sec. 3, there remain only paths for which the sign in front of the root does not change. We get the nonrelativistic Feynman principle (1) for the classical Lagrangian (3).

5. THE FEYNMAN PRINCIPLE FOR THE DIRAC EQUATION

We take the Dirac equation in the Majorana representation:

$$[\gamma_\nu(\partial/\partial x_\nu - iA_\nu) + \mu]\psi = 0; \quad \gamma_1 = -\sigma_3, \quad \gamma_2 = \sigma_1, \quad \gamma_3 = \sigma_3\sigma_1\rho_3\rho_1, \quad \gamma_4 = i\sigma_3\sigma_1\rho_1;$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

In addition to the usual representation, in which the matrices σ_3 and ρ_3 are diagonal (in this representation we shall denote the spinor index by the letter α), we take a representation in which ρ_1 and σ_1 are diagonal (and in which we denote the spinor index by β). The Feynman relation for an infinitely small time interval is obtained in this mixed representation by a simple generalization of Eq. (6) and has the form

$$R_{\alpha\beta}(x_\nu + \varepsilon_\nu | x_\nu) = \pm\frac{1}{2}\exp\{i(-\mu\varepsilon_s + A_\nu\varepsilon_\nu)\}, \qquad \varepsilon_\nu = \pm\varepsilon, \quad i\varepsilon_s = \pm\varepsilon, \tag{12}$$

Here ϵ is the change of the proper time.

The choise of signs is determined by the values of the indices α, β, i.e., the Dirac equation is replaced by the relation

$$\psi_\alpha(x_\nu) = \sum_\beta \eta_{\alpha\beta} \exp\{i(-\mu\varepsilon_s + A_\nu\varepsilon_\nu)\}\psi_\beta(x_\nu - \varepsilon_\nu), \quad \varepsilon_\nu = f(\alpha, \beta)\varepsilon, \quad f(\alpha, \beta) = \pm 1, \quad \eta_{\alpha\beta} = \pm 1/2.$$

The four values of the initial spinor indices and the four values of the final spinor indices give 16 possible combinations, and each combination prescribes a choice of the signs, i.e., a path of the electron from the center of the four-dimensional cube of side 2ϵ to one of the 16 vertices (space x, y, z, t if ϵ_s is prescribed, or x, y, z, s if ϵ_t is prescribed).

For finite time intervals $t - t' = N\epsilon$,

$$R(x_\nu, s \mid x'_\nu, s) = 4^{-N} \sum_{\varepsilon^k_\nu,\, \varepsilon^k_s} \pm e^{iS}, \qquad \sum \varepsilon^k_\nu = x_\nu - x'_\nu, \quad \sum \varepsilon^k_s = s - s', \quad S \to -\int \mu ds + \int A_\nu dx_\nu, \quad \varepsilon \to 0.$$

The sign of the path is the product of the signs of the successive displacements. As in the two-dimensional case, this sign is determined by the numbers of left and right turns (in the planes x, y and z, s). The expression (12) is interesting for the reason that not only does the path, i.e., the succession of coordinate values, determine the succession of spinor indices, but also conversely the succession of indices determines the path in space. One can renounce the latter requirement and seek the Feynman relation in the form

$$R_{\alpha\alpha'}(x_\nu + \varepsilon_\nu, x_\nu) = \sum_\gamma \eta_i(\alpha, \alpha', \gamma) \exp\{i(-\mu\varepsilon_s + A_\nu\varepsilon_\nu)\}, \quad \varepsilon_\nu = f(\alpha, \alpha', \gamma)\varepsilon. \tag{13}$$

The sum over γ denotes the sum over all displacements with prescribed α and α'. For each choice of the elementary displacements (i.e., each concrete meaning of γ) and each representation one can choose $f(\alpha, \alpha', \gamma)$ (i.e., the geometrical meaning of the indices) and the sign factor $\eta(\alpha, \alpha', \gamma)$, using the requirement that Eq. (13) go over into the Dirac equation for $\epsilon \to 0$.

6. CONCLUSION

Let us examine the physical and mathematical considerations which have made it possible to generalize the Feynman principle.

The first of these is a more precise interpretation of the meaning of relativistic wave functions. As has been shown, the relativistic wave function $\psi\alpha(x_\nu)$ gives not only the probability of the particle being at the point x_ν, but also certain indications about the motion of the particle through this point. This information is contained in the index α. All the paths that arrive at x_ν are divided into four classes, and $\psi_\alpha(x_\nu)$ means the sum over all paths of the type α.

Although the physical facts are simple, it is hard to write them down in the language of continuous paths. Therefore one has to regard space and time as discrete and correspondingly to alter the meaning of the symbol $\delta x(t)$ as compared with the use in Ref. 1 [cf. Eq. (1)]. The formulas become somewhat cumbersome but, as can be seen from Sec. 2, the infinitely multiple sums encountered here can be calculated relatively simply.

In Sec. 3 one more difference from Feynman's results appears. The domain of paths of integration (summation) in Eq. (7) turns out to be broader than the domain of comparison paths in the classical principle of least action. The form of the action is just the same: $-\int \mu\, ds + \int A_\nu dx_\nu$ (or $\int A_\alpha dx_\alpha$, $\alpha = 1, 2, 3, 4, 5$ in a five-dimensional notation). But in the principle of least action one considers only paths for which $ds^2 = dt^2 - d\mathbf{x}^2$ and $ds > 0$. In Eq. (7) one does not have this limitation and the "actual" classical picture (i.e., the picture for which the Feynman principle gives the Dirac equation) turns out to be symmetrical in relation to all five coordinates.

Finally there remains the antisymmetry of left and right turns considered in Sec. 2. This antisymmetry is obtained automatically if we replace the usual requirement on the Lagrangian—that it leave the principle of least action invariant—by the requirement of covariance of the Feynman principle.[1]

In conclusion I warmly thank V. Ia. Fainberg for direction and advice.

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NONRADIATIVE RECOMBINATION OF ELECTRONS AT IMPURITY CENTERS IN N-TYPE GERMANIUM

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Nonradiative ("phonon") recombination of a current carrier at a Coulomb impurity center in n-type germanium is investigated at helium temperatures. Interaction between an electron and acoustic vibrations of the lattice is taken into account by successive diagonalization of the original Hamiltonian by a unitary transformation. The tensor nature of the effective carrier mass and also the general expression for the deformation potential are taken into account in the calculations.

1. FORMULATION OF THE PROBLEM

The complex character of the energy bands in germanium and silicon, which has been revealed in cyclotron ("diamagnetic") resonance experiments and in certain other experiments, again raises the problem of calculating the energy levels and wave functions of the electron in an impurity center in these crystals. For n-type samples there exist experimental data concerning the existence of a "longitudinal" effective mass, μ_l, and a "transverse" effective mass, μ_t, of the electron. The tensor character of the effective mass has made it necessary to reexamine the old "scalar" model of impurity centers.[1,2] In these investigations, however, the interaction between the electron and the lattice vibration has not been taken into account. On the other hand, Deigen's[3] work has shown, that when one takes account of such interactions (deformation potential) for impurities in homopolar crystals, there appear in some cases comparatively deep levels, located below the bottom of the conduction band. Apparently, with respect to this so called "condenson" effect,[4] one encounters in Ge a case of weak binding. General expressions for the deformation potential in n-type germanium were obtained by Dumke,[5] who calculated, in the same paper, the scattering of current carriers in an ideal crystal by interaction with the lattice vibrations.

The purpose of the present work is to take into account the deformation potential for an electron in an impurity center by means of the weak-coupling approximation, and also to construct a corresponding theory for non-radiative recombination. It must be emphasized, that within the framework of the proposed method of calculation, the discrete and continuous spectra of electrons in impurity centers are considered together and thus the well known difficulties in this question[6] can be avoided.